Lecture 23 Outline:
Laplace Examples, Inverse, Rational Form

- **Announcements:**
  - **Reading:** “6: Laplace Transform” pp. 13-33, 55.5-56.5, 72
  - HW 8 posted, due next Wed. **Free 1-day extension**
  - OceanOne Robot Tour will be after class May 27 (11:30-12:20)
    - Lunch provided afterwards. Can arrange separate tour for those w/conflicts

- Laplace Transform Pairs/Properties (ppt only)
- LTI Systems Analysis using these Properties
- ROCs for Right/Left/Two-sided Signals
- Magnitude and Phase of FTs from Laplace
  - Bode Plots
- Inverse of Rational Laplace Transforms
Review of Last Lecture

- **Laplace Examples**
  - **Rect function:**
    \[ x_2(t) = \Pi\left(\frac{t}{2\tau}\right) = \begin{cases} 1 & |t| \leq \tau \\ 0 & |t| > \tau \end{cases} \]
    \[ \Rightarrow \quad \frac{e^{s\tau} - e^{-s\tau}}{s}, \quad \text{ROC} = \forall s \quad X_2(j\omega) = 2\tau \text{sinc}\left(\frac{\omega\tau}{\pi}\right) \]
  - **Left-Sided Real Exponential:**
    \[ -e^{-at}u(-t) \quad \Rightarrow \quad X_4(s) = \frac{1}{s+a}, \quad \text{Re}(s) < -a \quad X_4(j\omega) = \frac{1}{j\omega+a} \]
  - **Two-Sided Real Exponential**
    \[ x_5(t) = e^{b|t|}, \quad b \text{ real} \quad \Rightarrow \quad X_5(s) = \frac{1}{s-b} - \frac{1}{s+b} = \frac{2b}{(s-b)(s+b)} = \frac{2b}{s^2 - b^2} \]
    \[ \text{ROC} = \{\text{Re}(s) > b\} \cap \{\text{Re}(s) < -b\} \]
  - **Inverse Laplace transforms use complex analysis**
    - Can obtain inverse for rational Laplace transform using simple math (partial fraction expansion)
Review Cont’d

- **Rational Laplace Transforms**

\[
X(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \cdots + b_{M-1} s^{M-1} + b_M s^M}{a_0 + a_1 s + \cdots + a_{N-1} s^{N-1} + a_N s^N} = \frac{b_M (s - \beta_1)(s - \beta_2) \cdots (s - \beta_{M-1})(s - \beta_M)}{a_N (s - \gamma_1)(s - \gamma_2) \cdots (s - \gamma_{N-1})(s - \gamma_N)}
\]

- **Zeros and Poles:**
  - \(\beta\)s are zeros (where \(X(s)=0\)), \(\gamma\)s are poles (where \(X(s)=\infty\))
  - ROC cannot include any poles
  - If \(X(s)\) real, all zeros of \(X(s)\) are real or occur in complex-conjugate pairs. Same for the poles.
  - \(b_m/a_n\), zeros, poles, and ROC fully specify \(X(s)\)

- **Example: Two-sided Exponential**
  - Typo in Wed lecture: **no zeros in \(X_5(s)\)**

\[
X_5(s) = \frac{1}{s-b} - \frac{1}{s+b} = \frac{2b}{(s-b)(s+b)} = \frac{2b}{s^2 - b^2}
\]
**TABLE 9.1  PROPERTIES OF THE LAPLACE TRANSFORM**

<table>
<thead>
<tr>
<th>Section</th>
<th>Property</th>
<th>Signal</th>
<th>Laplace Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$x(t)$</td>
<td>$X(s)$</td>
<td>$R$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x_1(t)$</td>
<td>$X_1(s)$</td>
<td>$R_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$x_2(t)$</td>
<td>$X_2(s)$</td>
<td>$R_2$</td>
</tr>
<tr>
<td>9.5.1  Linearity</td>
<td></td>
<td>$a x_1(t) + b x_2(t)$</td>
<td>$aX_1(s) + bX_2(s)$</td>
<td>At least $R_1 \cap R_2$</td>
</tr>
<tr>
<td>9.5.2  Time shifting</td>
<td></td>
<td>$x(t - t_0)$</td>
<td>$e^{-st_0}X(s)$</td>
<td>$R$</td>
</tr>
<tr>
<td>9.5.3  Shifting in the $s$-Domain</td>
<td></td>
<td>$e^{s_0 t} x(t)$</td>
<td>$X(s - s_0)$</td>
<td>Shifted version of $R$ (i.e., $s$ is in the ROC if $s - s_0$ is in $R$)</td>
</tr>
<tr>
<td>9.5.4  Time scaling</td>
<td></td>
<td>$x(at)$</td>
<td>$\frac{1}{</td>
<td>a</td>
</tr>
<tr>
<td>9.5.5  Conjugation</td>
<td></td>
<td>$x^*(t)$</td>
<td>$X^<em>(s^</em>)$</td>
<td>$R$</td>
</tr>
<tr>
<td>9.5.6  Convolution</td>
<td></td>
<td>$x_1(t) * x_2(t)$</td>
<td>$X_1(s)X_2(s)$</td>
<td>At least $R_1 \cap R_2$</td>
</tr>
<tr>
<td>9.5.7  Differentiation in the Time Domain</td>
<td></td>
<td>$\frac{d}{dt} x(t)$</td>
<td>$sX(s)$</td>
<td>At least $R$</td>
</tr>
<tr>
<td>9.5.8  Differentiation in the $s$-Domain</td>
<td></td>
<td>$-tx(t)$</td>
<td>$\frac{d}{ds} X(s)$</td>
<td>$R$</td>
</tr>
<tr>
<td>9.5.9  Integration in the Time Domain</td>
<td></td>
<td>$\int_{-\infty}^{t} x(\tau)d(\tau)$</td>
<td>$\frac{1}{s} X(s)$</td>
<td>At least $R \cap {\text{Re}{s} &gt; 0}$</td>
</tr>
</tbody>
</table>

**Initial- and Final-Value Theorems**

- If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then
  \[ x(0^+) = \lim_{s \to \infty} sX(s) \]
  Applies to causal signals;
  Useful to check initial and final values of $x(t)$ without inverting $X(s)$

- If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then
  \[ \lim_{s \to \infty} x(t) = \lim_{s \to \infty} sX(s) \]
LTI Systems Analysis using these Properties

- **LTI Analysis using convolution property**

\[ y(t) = x(t) * h(t) \]
\[ H(s) = \mathcal{L}\{x(t) * h(t)\} \]
\[ Y(s) = H(s)X(s) \]

- **Equivalence of Systems (explored in HW)**

\[ g(t) * h(t) = g(t) + h(t) \]

\[ y(t) = x(t) + y(t) \]

\[ G(s) + H(s) \]

\[ \text{Follows from linearity of convolution and of the Laplace Transform} \]

\[ \text{Follows from commutivity of convolution and the convolution property of the Laplace Transform} \]
**TABLE 9.2 LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS**

<table>
<thead>
<tr>
<th>Transform pair</th>
<th>Signal</th>
<th>Transform</th>
<th>ROC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \delta(t) )</td>
<td>1</td>
<td>All ( s )</td>
</tr>
<tr>
<td>2</td>
<td>( u(t) )</td>
<td>( \frac{1}{s} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
<tr>
<td>3</td>
<td>(-u(-t))</td>
<td>( \frac{1}{s} )</td>
<td>( \Re{s} &lt; 0 )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{t^{n-1}}{(n-1)!} u(t) )</td>
<td>( \frac{1}{s^n} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
<tr>
<td>5</td>
<td>( -\frac{t^{n-1}}{(n-1)!} u(-t) )</td>
<td>( \frac{1}{s^n} )</td>
<td>( \Re{s} &lt; 0 )</td>
</tr>
<tr>
<td>6</td>
<td>( e^{-\alpha t} u(t) )</td>
<td>( \frac{1}{s + \alpha} )</td>
<td>( \Re{s} &gt; -\alpha )</td>
</tr>
<tr>
<td>7</td>
<td>( -e^{-\alpha t} u(-t) )</td>
<td>( \frac{1}{s + \alpha} )</td>
<td>( \Re{s} &lt; -\alpha )</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(t) )</td>
<td>( \frac{1}{(s + \alpha)^n} )</td>
<td>( \Re{s} &gt; -\alpha )</td>
</tr>
<tr>
<td>9</td>
<td>( -\frac{t^{n-1}}{(n-1)!} e^{-\alpha t} u(-t) )</td>
<td>( \frac{1}{(s + \alpha)^n} )</td>
<td>( \Re{s} &lt; -\alpha )</td>
</tr>
<tr>
<td>10</td>
<td>( \delta(t - T) )</td>
<td>( e^{-sT} )</td>
<td>All ( s )</td>
</tr>
<tr>
<td>11</td>
<td>( [\cos \omega_0 t] u(t) )</td>
<td>( \frac{s}{s^2 + \omega_0^2} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
<tr>
<td>12</td>
<td>( [\sin \omega_0 t] u(t) )</td>
<td>( \frac{s}{s^2 + \omega_0^2} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
<tr>
<td>13</td>
<td>( [e^{-\alpha t} \cos \omega_0 t] u(t) )</td>
<td>( \frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2} )</td>
<td>( \Re{s} &gt; -\alpha )</td>
</tr>
<tr>
<td>14</td>
<td>( [e^{-\alpha t} \sin \omega_0 t] u(t) )</td>
<td>( \frac{\omega_0}{(s + \alpha)^2 + \omega_0^2} )</td>
<td>( \Re{s} &gt; -\alpha )</td>
</tr>
<tr>
<td>15</td>
<td>( u_n(t) = \frac{d^n \delta(t)}{dt^n} )</td>
<td>( s^n )</td>
<td>All ( s )</td>
</tr>
<tr>
<td>16</td>
<td>( u_{-n}(t) = u(t) \ast \cdots \ast u(t) )</td>
<td>( \frac{1}{s^n} )</td>
<td>( \Re{s} &gt; 0 )</td>
</tr>
</tbody>
</table>

(Similar to CTFT pairs
Replace \( j\omega \) with \( s \), check ROC)
ROC for “Sided” Signals

- **Right-sided:** $x(t) = 0$ for $t < a$ for some $a$
  - ROC is to the right of the rightmost pole
  - Example: RH exponential
  $$L[x(t)] = X(s) = \int_{a}^{\infty} x(t)e^{-st} \, dt$$

- **Left-sided:** $x(t) = 0$ for $t > a$ for some $a$
  - ROC is to the left of the leftmost pole
  - Example: LH exponential
  $$L[x(t)] = X(s) = \int_{a}^{-\infty} x(t)e^{-st} \, dt$$

- **Two-sided:** neither right or left sided
  - Can be written as sum of RH and LH sided signals
  - ROC is a vertical strip between two poles,
  - 2-sided exponential
Magnitude/Phase of Fourier from Laplace and Bode Plots

- **Magnitude/Phase of Fourier from Laplace**
  - Given Laplace in rational form with $j\omega$ in ROC
    
    \[ H(s) = \frac{\prod_{k=1}^{M} (s - \beta_k)}{\prod_{k=1}^{N} (s - \gamma_k)} \]
    
    \[ H(j\omega) = \frac{\prod_{k=1}^{M} (j\omega - \beta_k)}{\prod_{k=1}^{N} (j\omega - \gamma_k)} \]
    
    \[ |H(j\omega)| = \left| \frac{\prod_{k=1}^{M} |j\omega - \beta_k|}{\prod_{k=1}^{N} |j\omega - \gamma_k|} \right| \]
    
    \[ \angle H(j\omega) = \angle \left( \frac{\prod_{k=1}^{M} (j\omega - \beta_k)}{\prod_{k=1}^{N} (j\omega - \gamma_k)} \right) + \sum_{k=1}^{M} \angle(j\omega - \beta_k) - \sum_{k=1}^{N} \angle(j\omega - \gamma_k) \]
    
    - Can obtain magnitude and phase from individual components using geometry, formulas, or Matlab

- **Bode Plots: show magnitude/phase on dB scale**
  - Plots $10\log_{10}|H(j\omega)|^2 = 20\log_{10}|H(j\omega)|$ vs. $\omega$ (dB)
  - Can plot exactly or via straight-line approximation
  - Poles lead to 20dB per decade decrease in Bode plot, Zeros lead to 20 dB per decade increase
  - Frequency $\omega$ in rad/s is plotted on a log scale for $\omega \geq 0$ only
Example: First-order LPF

\[ h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t) \quad \leftrightarrow \quad H(s) = \frac{1}{1 + s \tau} = \frac{1}{\tau \left( s - \left( -\frac{1}{\tau} \right) \right)} \]

\( \tau > 0, \tau \text{ real} \)

\[ RC = \tau \]

\[ H(j\omega) = \frac{1}{1 + \tau j\omega} = \frac{1}{\tau(j\omega - \gamma)} \]

\( \text{ROC} = \{ \sigma > -1/\tau \} \)
Bode Plot for 1\textsuperscript{st} Order LPF

\begin{align*}
\text{0 dB for } \omega < \frac{1}{\tau} \\
\text{0 rad for } \omega < \frac{0.1}{\tau} \\
\text{\(\frac{-\pi}{2}\) rad for } \omega > \frac{0.1}{\tau} \\
\text{\(\frac{-\pi}{4}\) rad for } \omega = \frac{1}{\tau} \\
\text{\(\frac{-\pi}{2}\) rad for } \omega > \frac{10}{\tau} \\
\end{align*}
**Inversion of Rational Laplace Transforms**

\[ X(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \cdots + b_{M-1} s^{M-1} + b_M s^M}{a_0 + a_1 s + \cdots + a_{N-1} s^{N-1} + a_N s^N} \]

- **Extract the Strictly Proper Part of** \( X(s) \)
  - If \( M < N \), \( X(s) = \tilde{X}(s) \) is strictly proper, proceed to next step
  - If \( M \geq N \), perform long division to get \( X(s) = D(s) + \tilde{X}(s) \), where
    \[ D(s) = d_{M-N} s^{M-N} + d_{M-N-1} s^{M-N-1} + \cdots + d_1 s + d_0 \]
  - Invert \( D(s) \) to get time signal:
    \[ d(t) = d_{M-N} \delta^{(M-N)}(t) + d_{M-N-1} \delta^{(M-N-1)}(t) + \cdots + d_1 \delta^{(1)}(t) + d_0 \delta^{(0)}(t), \]
  - Follows from \( \delta^{(n)}(t) = \frac{d^n \delta(t)}{dt^n} \) and \( \frac{d^n z(t)}{dt^n} \leftrightarrow s^n Z(s) \)
  - The second term \( \tilde{X}(s) = \tilde{B}(s)/A(s) \) is strictly proper

- **Perform a partial fraction expansion:**
  \[ \tilde{X}(s) = \frac{\tilde{B}(s)}{A(s)} = \frac{1}{a_N} \frac{\tilde{B}(s)}{ \prod_{k=p+1}^{N} (s - \gamma_k) } \]
  \[ \tilde{X}(s) = \sum_{m=1}^{p} \frac{A_m}{(s - \gamma'_m)} + \sum_{k=p+1}^{N} \frac{A_k}{s - \gamma_k} \]
  Obtain coefficients via residue method

- **Invert partial fraction expansion term-by-term**
  - For right-sided signals: \( \tilde{x}(t) = \sum_{m=1}^{p} A_m \frac{t^{m-1}}{(m-1)!} e^{\gamma'_t} u(t) + \sum_{k=p+1}^{N} A_k e^{\gamma_k t} u(t) \)
Main Points

- Properties of Laplace transforms similar to those of Fourier transforms, with similar proofs. Must check ROC.

- Convolution and other properties of Laplace allow us to study input/output relationship LTI systems
  - Can also study LTI systems in series or parallel

- One-Sided signals have one-sided ROCs. Two sided signals have strips as ROCs

- Can determine magnitude and phase of Fourier transform easily from rational form of Laplace
  - Bode plots show magnitude/phase on dB scale
  - Poles change slope by \(-20\text{dB/decade}\), zeros change slope by \(20\text{dB per decade}\)

- Invert rational Laplace transforms using partial fraction expansion