Lecture 26 Outline:
LTI Systems Analysis and Feedback

- Announcements:
  - Last HW will be posted today, due Thu June 7, no late HWs
  - Final exam announcements on next slide

- LTI Systems Described by Difference Equations

- Infinite Impulse Response due to Feedback

- Realization of IIR Systems

- Example: 2\textsuperscript{nd} order Filter

- Feedback in discrete LTI Systems
Final Exam Details

- Time/Location: Monday June 11, 3:30-6:30 in this room. Food afterwards.
- Open book and notes – you can bring any written material you wish to the exam. Calculators not allowed. PDF browsing devices allowed.
- Covers all class material; emphasis on post-MT material (lectures 14 on)
  - See lecture ppt slides for material in the reader that you are responsible for
- Practice final will be posted by Friday, worth 25 extra credit points for “taking” it (not graded).
  - Can be turned in any time up until exam, Solns given when you turn in your answers
- My final review on W 6/6 in class. Georgia Th 6/7 4-6pm (review + OHs)
- OHs next week and before final:
  - Me: 6/4, Mon, 3-4; 6/6, Wed, 12-1:30 (here); 6/8, Fri, by appt. (request by Thu 5pm)
  - Regular Tuesday section and TA OHs next week plus Th 6/7 ~5-6pm (Georgia) Sun, 6/10, 12-2pm (John), Mon 6/11, 12-2pm (Malavika)
Review of Last Lecture

- **Z-transforms of sided signals**
  - Right-sided, ROC is *outside* circle for largest pole
  - Left-sided, ROC is *inside* circle for smallest pole
  - Two-sided, ROC is *concentric* circle with no poles

- **Inversion of Rational z-Transforms**
  
  \[
  X(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{-(M-1)} + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{-(N-1)} + a_N z^{-N}} = \frac{b_0}{a_0} \frac{(1 - \beta_1 z^{-1})(1 - \beta_2 z^{-1}) \cdots (1 - \beta_{M-1} z^{-1})(1 - \beta_M z^{-1})}{(1 - \gamma_1 z^{-1})(1 - \gamma_2 z^{-1}) \cdots (1 - \gamma_{N-1} z^{-1})(1 - \gamma_N z^{-1})}
  \]

  \[
  = \frac{b_0}{a_0} z^{N-M} \frac{(z - \beta_1)(z - \beta_2) \cdots (z - \beta_{M-1})(z - \beta_M)}{(z - \gamma_1)(z - \gamma_2) \cdots (z - \gamma_{N-1})(z - \gamma_N)}
  \]

  Same format as Laplace:
  - \(N>M, N-M\) zeroes at \(z=0\)
  - \(N<M, M-N\) poles at \(z=0\)

- **Extract the Strictly Proper Part of \(X(z)\):**
  \(X(z) = D(z) + \tilde{X}(z)\)

  \[
  D(z) = d_{M-N} z^{M-N} + d_{M-N-1} z^{M-N-1} + \cdots + d_1 z + d_0 \quad \leftrightarrow \quad d[n] = d_{M-N} \delta[n-(M-N)] + \cdots + d_1 \delta[n-1] + d_0 \delta[n]
  \]

- **Partial fraction expansion:**
  \(\tilde{X}(z) = \frac{\tilde{B}(z)}{A(z)} = \frac{1}{a_N} \frac{\tilde{B}(z)}{\prod_{k=1}^{N} (z - \gamma_k)}\)

  \[
  \tilde{X}(z) = \sum_{m=1}^{n} \frac{A_m}{(z - \gamma_m)^m} + \sum_{k=p+1}^{N} \frac{A_k}{z - \gamma_k}
  \]

Obtain coefficients via residue method
Review Continued

LTI Systems Analysis using z-Transforms

- LTI Analysis using convolution property

\[ y[n] = x[n] * h[n] \]

\[ Y[z] = H[z]X[z] \]

\[ H[z] \] called the transfer function of the system

ROC is at least \( \text{ROC}_x \cap \text{ROC}_h \)

- Equivalence of Systems same as for Laplace


- Causality and Stability in LTI Systems

  - System is causal if \( h[n]=0, \) \( n<0 \)
    - ROC outside circle associated with largest pole since \( h[n] \) right sided
  - Causal system (BIBO) stable if bounded inputs yield bounded outputs
  - BIBO stable implies that \( h[n] \) absolutely summable \( \Rightarrow H(e^{i\Omega}) \) exists
  - \( H(z) \) is stable \( \Leftrightarrow H(z) \) strictly proper and poles inside unit circle, i.e. \( r=1 \in \text{ROC} \) (see Reader)
LTI Systems Described by Difference Equations

- DT systems implemented with delay/amplifier elements

- A system with output feedback generally has infinite impulse response

- LTI Systems described by differential equations

\[
\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \rightarrow \left( \sum_{k=0}^{N} a_k z^{-k} \right) \cdot Y(z) = \left( \sum_{k=0}^{M} b_k z^{-k} \right) \cdot X(z) \rightarrow H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}
\]

- Can write in terms of output at time \( n \)

\[
y[n] = \frac{1}{a_0} \left( \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k] \right)
\]
Direct Form Implementation of IIR Filters/Systems $H(z)$

Next lecture we will discuss IIR filter design. Approximates continuous time filters with lower complexity than FIR designs.
Example: Second Order System with Feedback

- **Z-Transform**

\[
H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}} = \frac{1}{(1 - re^{j\theta} z^{-1})(1 - re^{-j\theta} z^{-1})}, \quad |z| > r
\]

- **DTFT:** \( H(e^{j\Omega}) = \frac{1}{1 - 2r \cos \theta e^{-j\Omega} + r^2 e^{-j2\Omega}} \)
  - Exists when \( r < 1 \)
  - Lowpass filter for \( \theta = 0 \)
  - Bandpass filter for \( \theta = \pi / 2 \)
  - Highpass filter for \( \theta = \pi \)

- **Impulse Response (LPF: \( \theta = 0 \))**

\[
H(z) = \frac{1}{(1 - rz^{-1})^2} \quad \leftrightarrow \quad h[n] = [(n + 1)r^n]u[n]
\]
Feedback in discrete LTI Systems

- **Motivation for Feedback**
  - Can make an unstable system stable; make a system less sensitive to disturbances; make it closer to ideal
  - Can have negative effect: make a stable system unstable

- **Transfer Function** $T(z)$ of Feedback System:
  \[ Y(z) = G(z)E(z); \quad E(z) = X(z) \pm Y(z)H(z) \]
  \[ T(z) = \frac{Y(z)}{X(z)} = \frac{G(z)}{1 \mp G(z)H(z)} \]

- **Example: Population Growth:**
  \[ y[n] = 2y[n-1] + x[n] - r[n] \]
  - Pole at $z = 2-2\beta$, Stable for $0.5 < \beta < 1.5$
  \[ T(z) = \frac{1}{1 - 2(1 - \beta)z^{-1}} \]
Main Points

- LTI systems/filters described with difference equations
  - Output feedback leads to infinite impulse response systems

- Systems of difference equations easily characterized using $z$-transforms
  - IIR systems can be easily implemented in direct form based on the difference equations
  - IIR filter design generally lower complexity than FIR filter design to approximate a given continuous time filter

- Feedback stabilizes unstable systems; obtains desired transfer function; can make stable system unstable
  - Easy to analyze systems with feedback using $z$-transforms