Lecture 27 Outline:
Systems Analysis w/ Z Transforms, Course Review

- **Announcements:**
  - Reading: “8: z-transforms” pp. 36-50 (no inverse or unilateral z transforms)
  - HW 9 posted, due 6/3 midnight (no late HWs);
  - Final exam announcements on next slide

- **LTI Systems Analysis using Z-Transforms**

- **Feedback in LTI Systems**

- **Example: 2\textsuperscript{nd} order system with feedback**

- **Course Review**
Final Exam Details

- Time/Location: **Tuesday June 7, 8:30-11:30am in this room.**
- Open book and notes – you can bring any written material you wish to the exam. Calculators and electronic devices not allowed.
- Covers all class material; emphasis on post-MT material (lectures 15 on)
  - See lecture ppt slides for material in the reader that you are responsible for
- Practice final will be posted by tomorrow, worth 25 extra credit points for “taking” it (not graded).
  - Can be turned in any time up until you take the exam
  - Solutions given when you turn in your answers
  - In addition to practice MT, we will also provide additional practice problems/solns
- Instead of final review, we will provide extra OHs for me and the TAs to go over course material, practice MT, and practice problems
  - Alon: Friday 6/3 5-6 PM, Monday 6/6 5-6 PM; Mainak: Thursday 6/2 3-4 PM, Monday 6/6 3-4 PM; Jeremy: Thursday 6/2 1-2 PM, Monday 6/6 7-8 PM
Review of Last Lecture

- **Bilateral Z Transform**
  
  Defn: \( Z\{x[n]\} = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \), \( z = |z| e^{j\Omega} \); exists if \( \sum_{n=-\infty}^{\infty} x[n]r^{-n} < \infty \)

- **Relation with DTFT**: \( X(z)_{r=|z|=1} = X(e^{j\Omega}) = DTFT\{x[n]\} \)

- **Region of Convergence (ROC):**
  - All \( z \) s.t. \( X(z) \) exists, depends only on \( r \), bounded by circles
  - ROCs for sided signals are sided

- **Rational Z-Transforms**:
  
  - Written in terms of \( z \) or \( z^{-1} \) (\( z \) in ppt slides, \( z^{-1} \) in lecture, \( z^{-1} \) more common)
  
  \[
  X(z) = \frac{b_0 + b_1z^{-1} + \cdots + b_{M-1}z^{-(M-1)} + b_Mz^{-M}}{a_0 + a_1z^{-1} + \cdots + a_{N-1}z^{-(N-1)} + a_Nz^{-N}} = \frac{b_0 (1 - \beta_1z^{-1})(1 - \beta_2z^{-1})\cdots(1 - \beta_{M-1}z^{-1})(1 - \beta_Mz^{-1})}{a_0 (1 - \gamma_1z^{-1})(1 - \gamma_2z^{-1})\cdots(1 - \gamma_{N-1}z^{-1})(1 - \gamma_Nz^{-1})}
  \]

  - Invert rational z-transforms using partial fraction expansion

- **Properties and transform pairs of z-transforms**
LTI Systems Analysis using z-Transforms

- LTI Analysis using convolution property
  \[ x[n] \rightarrow h[n] \rightarrow y[n] \]
  \[ y[n] = x[n] * h[n] \]
  \[ H(z) \]

- Equivalence of Systems same as for Laplace
  - Systems G[z] and H[z] in Parallel: \( T[z] = G[z]H[z] \)
  - Systems G[z] and H[z] in Series: \( T[z] = G[z] + H[z] \)

- Causality and Stability in LTI Systems
  - System is causal if \( h[n] = 0, \ n < 0 \), so is \( h[n] \) right-sided; has its ROC outside circle associated with largest pole.
  - System BIBO stable if bounded inputs yield bounded outputs; true iff \( h[n] \) absolutely summable \( \Rightarrow H(e^{j\Omega}) \) exists \( \Rightarrow \) poles inside unit circle

- LTI Systems described by differential equations
  \[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \]
  \[ \left( \sum_{k=0}^{N} a_k z^{-k} \right) y(z) = \left( \sum_{k=0}^{M} b_k z^{-k} \right) x(z) \]
  \[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)} \]
Feedback in LTI Systems

- **Motivation for Feedback**
  - Can make an unstable system stable, make a system less sensitive to disturbances, closer to ideal
  - Can have negative effect: make a stable system unstable

- **Transfer Function** $T(z)$ of Feedback System:
  $Y(z) = G(z)E(z); \ E(z) = X(z) \pm Y(z)H(z)$
  $T(z) = \frac{Y(z)}{X(z)} = \frac{G(z)}{1 \mp G(z)H(z)}$

- **Example: Population Growth:** $y[n] = 2y[n-1] + x[n] - r[n]$
  - Stable for $\beta < .5$

$$T(z) = \frac{1}{1 - 2(1 - \beta)z^{-1}}$$
Example: Second Order System with Feedback

- **Z-Transform**
  
  \[ H(z) = \frac{1}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}} = \frac{1}{(1 - re^{j\theta} z^{-1})(1 - re^{-j\theta} z^{-1})}, \ |z| > r \]

- **DTFT:** \[ H(e^{j\Omega}) = \frac{1}{1 - 2r \cos \theta e^{-j\Omega} + r^2 e^{-j2\Omega}} \]
  - Exists when \( r < 1 \)
  - Lowpass filter for \( \theta = 0 \)
  - Bandpass filter for \( \theta = \pi/2 \)
  - Highpass filter for \( \theta = \pi \)

- **Impulse Response (LPF: \( \theta = 0 \))**
  
  \[ H(z) = \frac{1}{\left(1 - rz^{-1}\right)^2} \quad \leftrightarrow \quad h[n] = [(n+1)r^n]u[n] \]
Main Points

- Convolution and other properties of z-transforms allow us to study input/output relationship of LTI systems
  - A causal system with $H(z)$ rational is stable if & only if all poles of $H(z)$ lie inside unit circle (all poles have $|z|<1$)
  - ROC defined implicitly for causal stable LTI systems

- Systems of difference equations easily characterized using z-transforms

- Feedback stabilizes unstable systems; obtains desired transfer function