Additional Practice Problems for Final Exam

1. Linear and circular convolution
In this problem we explore a new method to get the circular convolution of two sequences \( X = \{x[n]\}_{n=0}^{N-1} \) and \( H = \{h[n]\}_{n=0}^{M-1} \) from their linear convolution. Assume \( N > M \).

Define a new sequence \( X_p = \{x_p[n]\}_{n=-M+1}^{N-1} \) defined so that

\[
x_p[n] = \begin{cases} x[n], & \text{if } n \geq 0 \text{ and } n < N \\ 0, & \text{otherwise} \end{cases}
\]

So this is a zero-padded version of \( x[n] \) but with zeros at the beginning of the sequence, for \(-M+1 \leq n < 0\).

Let \( y[n] = x_p[n] * h[n] \) and define \( y'[n] = \begin{cases} y[n], & \text{if } N \geq n \geq M \\ y[n] + y[N+n], & \text{if } 0 \leq n \leq M \end{cases} \).

Show that \( y'[n] = h[n] \odot x[n] \) for \( n \in [0, N-1] \).

2. Discrete Fourier transforms
Consider a periodic signal \( \tilde{x}[n] \) with period \( N = 4 \) and the corresponding one-period signal \( x[n] \) with duration \( N = 4 \).

These signals can be described by the discrete Fourier series and discrete Fourier transform:

\[
\tilde{x}[n] \leftrightarrow \tilde{X}[k] \quad \text{and} \quad x[n] \leftrightarrow X[k].
\]

(a) (10 points) Describe all the symmetries present in the time signal and their implications for the Fourier description. You may consider either \( \tilde{x}[n] \leftrightarrow \tilde{X}[k] \) or \( x[n] \leftrightarrow X[k] \).

(b) (15 points) Give an expression for the DFT \( X[k] \). To earn full credit, your answer should be purely real or purely imaginary (the symbol \( j \) should appear at most once). You may find it easy to compute the DTFT \( X(e^{j\Omega}) \) and then sample it to obtain \( X[k] \).
(c) (5 points) Sketch $X[k]$ vs. $k$ for $0 \leq k \leq 3$. It is easiest to plot the real and imaginary parts (whichever is nonzero), rather than the magnitude and phase.

3. **Limits:** For a causal $x(t)$ whose transform is given by $X(s)=1/(s^2+2s)$, evaluate $\lim_{t \to \infty} x(t)$

4. **Z transform.**

We are given the following information about a signal $x[n]$ and its Z transform $X(z)$:

(i) $x[n]$ is real and right-sided.

(ii) $X(z)$ has exactly two zeros, both at the origin.

(iii) $X(z)$ has exactly two poles.

(iv) $X(z)$ has a pole at $z = \gamma_1 = \frac{1}{\sqrt{2}} e^{\frac{j\pi}{4}}$.

(v) $X(1) = 2$.

(a) Find an expression for $X(z)$ and specify its region of convergence.

(b) Find an expression for $x[n]$. To obtain full credit, your answer should be purely real. *Hint:* the coefficients in $X(z)$ have been chosen so that $X(z)$ can be inverted using two Z transforms appearing in transform tables, so no partial-fraction expansion is necessary. Nevertheless, the correct answer can be obtained using partial-fraction expansion.

5. **Discrete-time Systems Characterized by Difference Equations:** A causal LTI system is described by the difference equation:

$$y[n] = y[n-1] + y[n-2] + x[n-1]$$

(a) Find the system function $H(z) = Y(z)/X(z)$ for this system. Plot the poles and zeros of $H(z)$ and indicate the region of convergence.

(b) Find the unit sample response of the system.

(c) You should have found the system to be unstable. Find a stable (noncausal) unit sample response that satisfies the difference equation.

6. **Bilinear Z transform filter design.**

We start with a continuous-time first-order lowpass filter with impulse response and transfer function

$$h_c(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t) \leftrightarrow H_c(s) = \frac{1}{1+\frac{s}{\tau}}.$$
We choose a sampling rate $1/T$ and design a discrete-time filter $h[n] \leftrightarrow H(z)$ whose transfer function satisfies

$$H(z) = H_c(s) = \frac{2(1-z^{-1})}{T(1+z^{-1})}.$$ 

It is helpful to define

$$b = \frac{T - 2\tau}{T + 2\tau}.$$ 

(a) Give an expression for the transfer function $H(z)$. 
(b) Give an expression for the impulse response $h[n]$. 
(c) Sketch the poles and zeros of $H(z)$. 
(d) Sketch the magnitude response $|H(e^{j\omega})|$, assuming $T = 1$, $\tau = 2$, $b = 0.6$. 
