

## So you say “Bring on the SPAM?”



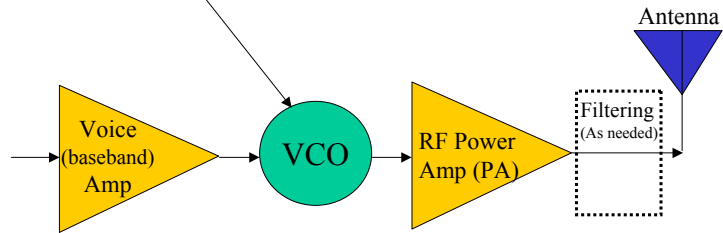
### Last Time's Lecture:

- Warm-ups about **Transmitters**
- **Angle Modulation**--> **FM** & **PM**
- How to get Modulation--> **VCO**
- Introduction to **Oscillators:**
  - ✓ Feedback Perspective
  - ✓ “Timing-based” (I.e. **566**)

### Today's Lecture:

- Wideband FM (Handout **#6**)
- Colpitts Oscillator (Handout **#5**)
- Tuned Circuits (first pass--Handout **#7**)

Colpitts Oscillator, combined with a “Varactor”  
(Variable Reactor=Variable Capacitance) gives you Direct FM  
Modulation (vs. Heterodyned FM)



# Frequency/Phase Mod.

## Frequency/Phase Modulation

Generally can formulate as **Angle Modulation**

$$v_o(t) = A \cos \Theta(t)$$

$$= A \cos(\omega t + \phi)$$

Frequency      Phase

(FM)            (PM)

Both are important and used in practice; let's do FM first

$$\omega = d\Theta/dt \quad (1)$$

Instantaneous frequency is

$$f = f_c + K_o v_m(t) \quad (2)$$

Equating (1) and (2)

$$d\Theta/dt = 2\pi f_c + 2\pi K_o v_m(t)$$

This slide basically introduces the overall concept of Angle Modulation and the sub-classes of FM and PM.

Basically, trigonometric functions only understand angles (0-360° or expressed in radians 0-2 $\pi$ )

For simplicity, let's break this into an " $\omega t$ " term and a phase term; given these two components, we can either modulate Freq. or Phase.

When we talk about "frequency" we are actually talking about the time derivative of the ANGLE

The instantaneous frequency can be thought of as the "nominal" carrier frequency + a time-dependent term.

Putting the two concepts (ways of formulating the problem) together we get the final expression.

However, to turn it back into an "angle" that the cosine understands, we need to integrate (next page)

Then integrating with respect to time

$$\Theta(t) = 2\pi f_c t + \Theta_0 + 2\pi K_o \int v_m(t) dt$$

Assuming  $v_m(t) = v_p \cos(2\pi f_m t)$   
 (and letting  $\Theta_0 = 0$  & integration constant = 0)

$$\Theta(t) = 2\pi f_c t + \frac{\overbrace{K_o v_p}^{\Delta f_c}}{f_m} \sin(2\pi f_m t)$$

$$v_o(t) = A \cos\left( \underbrace{2\pi f_c t}_a + \underbrace{m_f \sin(2\pi f_m t)}_b \right)$$

$$m_f \equiv \frac{K_o v_p}{f_m}$$

See Trig. Identity  
 on next Slide...

The above integration gives us the “angle” that the cosine (or other “trig” function) wants.

Now, applying a specific modulating cosine function, and dropping the other constants, we get the simplified  $\Theta(t)$  expression.

Note, the cosine becomes a sine due to the integration.

We also define how  $f_c$  gets changed by  $\Delta f_c$  and we actually call this term the modulation coefficient  $m_f$

Finally, if we put the “angle” back into the original trig. function, we get the last equation with  $\cos(a+b)$  where “a” is the carrier frequency and “b” relates to the changes in that frequency due to the modulation

Now, we get into some rather heavy (but you don’t have to sweat the derivations:) math...

**MORE FROM Handout #6** Narrow-band vs Wide-band FM

Narrow Band FM (NBFM) is defined such that  $m_f < 0.25$  (radians)  
 Generally, looking at the previous equation for  $v_O$ :

$$A \cos(a + b) \equiv A[\cos(b) \cdot \cos(a) - \sin(b) \cdot \sin(a)]$$

$$v_{out}(t) = A[\underbrace{\cos(m_f \sin \omega_m t) \cdot \cos(\omega_c t)}_{\text{red dashed line}} - \sin(m_f \sin \omega_m t) \cdot \sin(\omega_c t)]$$

For NBFM using small-angle formulae  $\cos(x) \sim 1$  and  $\sin(x) \sim x$

$$v_{out}(t) = A[\underbrace{\cos(\omega_c t) - m_f \sin \omega_m t \cdot \sin(\omega_c t)}_{\text{bracket}}]$$

Wow, the result looks like AM (...sort of...)

We start by defining Narrowband (NB) versus Wideband (WB) Frequency Modulation (FM) in terms of how big (or small)  $m_f$  is (units of radians)

The logic of using this definition becomes clear as we look at what happens to the Bessel functions versus this argument.

Again, starting from the trig. equation with argument (a+b) where “a” and “b” are different frequencies, when we write this out (as shown) it looks pretty messy...

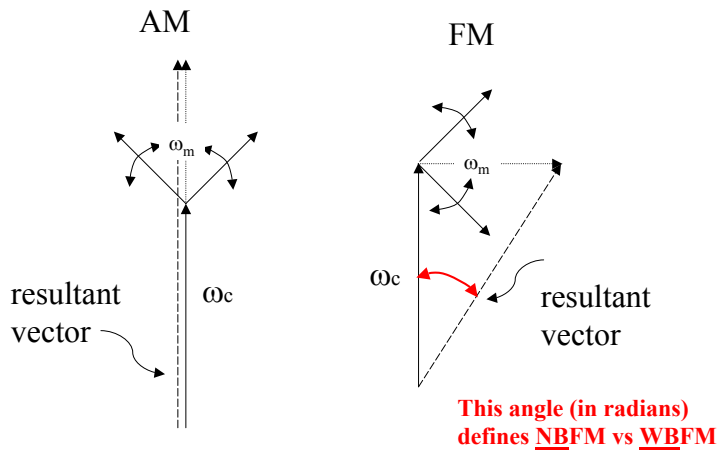
But, for NBFM we use the  $\cos(x) \sim 1$ ,  $\sin(x) \sim x$  for small (x) arguments, to simplify the expression.

We also note that this equation looks a lot like what we’ve done earlier for AM.

The key difference is that the first term is cos (not sin) which is at a “quadrature” with the  $\sin(\omega_m t)\sin(\omega_c t)$  term.

In the next slide we’ll see graphically what this means...

## Graphical View...AM vs FM



This figure summarizes AM versus FM in terms of “vector diagrams”

You have to imagine the dashed line vector (the resultant vector) as spinning around ( $0 \rightarrow 2\pi$ ) in the time domain.

For AM this “spinning vector” gets longer and shorter based on the two side arrows moving back and forth

[I guess I’ve got to “wave my hands” with a pointer to make this point]

For FM the two smaller arrows are in “quadrature” to the carrier frequency.

Now, as they “wiggle” the “resultant” vector’s angle is shifted as it spins

Bottom-line:

AM--arrow gets longer and shorter with constant angular rotation

FM--arrow experiences changed phase with respect to spinning vector (the length of the vector is CONSTANT)

## Wideband FM (or just FM... $m_f > 0.25$ )

$$\begin{aligned}v_o(t) &= \cos(\omega_c t + m_f \sin \omega_m t) \\ &= \cos \omega_c t \cdot \cos(m_f \sin \omega_m t) - \sin \omega_c t \cdot \sin(m_f \sin \omega_m t)\end{aligned}$$

Without using small angles, the result is Bessel Functions!

$$\begin{aligned}v_o|_{WBFM}(t) &= J_0(m_f) \cos \omega_c t + \\ &J_1(m_f) [\cos(\omega_c + \omega_m)t - \cos(\omega_c - \omega_m)t] + \\ &J_2(m_f) [\cos(\omega_c + 2\omega_m)t + \cos(\omega_c - 2\omega_m)t] + \\ &J_3(m_f) [\cos(\omega_c + 3\omega_m)t - \cos(\omega_c - 3\omega_m)t] + \\ &J_4(m_f) [\cos(\omega_c + 4\omega_m)t + \cos(\omega_c - 4\omega_m)t] + \dots\end{aligned}$$

As stated from the first line (and comments earlier), for Narrow Band FM the modulation factor  $m_f$  is sufficiently small that “small angle” approximations for sine and cosine reduce the equations to nearly equivalent to AM (with the exception that there is a “quadrature” between the carrier and modulating signals)

For Wide Band FM, we lift these simplifying approximations.

Hence...

We have to consider cosine and sine functions with additional trig. Functions as the arguments.

Fortunately, there are elegant solutions (as infinite series) based on the Bessel Functions.

Basically, the result is a set of PRODUCT terms that in turn will result in sum and difference terms in  $\omega_c \pm n\omega_m$  where “n” takes on integer values.

## Reminders about the Bessel Equation (etc.)

$$x^2 y'' + xy' + (x^2 - p^2)y = 0$$

$$J_0(x) = \sum_0^{\infty} (-1)^k \frac{x^{2k}}{2^{2k} (k!)^2}$$

$$J_1(x) = \sum_0^{\infty} (-1)^k \frac{x^{2k+1}}{2^{2k+1} k!(k+1)!}$$

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

This slide goes all the way back to the differential equations that result in the Bessel functions.

I really don't have much to say additionally about this point.

(TBD)

## Related (useful) Trigonometric Formulae

$$\cos(m \cos x) = J_0(m) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(m) \cos(2kx)$$

$$\cos(m \sin x) = J_0(m) + 2 \sum_{k=1}^{\infty} J_{2k}(m) \cos(2kx)$$

$$\sin(m \cos x) = 2 \sum_{k=0}^{\infty} (-1)^k J_{2k+1}(m) \cos(2k+1)x$$

$$\sin(m \sin x) = 2 \sum_{k=0}^{\infty} J_{2k+1}(m) \sin(2k+1)x$$

This slide shows that for each of the respective terms in the expansion for FM, that there are corresponding Bessel functions (and infinite series expansions) that correspond to them.

Clearly, based on this set of  $\cos \cdot \cos$ ,  $\cos \cdot \sin$ ,  $\sin \cdot \cos$  and  $\sin \cdot \sin$  cases and the overall equation set for the solution for  $v_o(t)|_{\text{WBFM}}$ , there is a final set of  $J_k$  and “sum and differences” of terms with increasing “m”

The next slides show this graphically.

My own conceptualization of the problem is best expressed in terms of the following (graphical versus mathematical) formulations...

**TABLE 9-1** Bessel Functions of the First Kind,  $J_n(m_f)$

$m_f$	$J_0$	$J_1$	$J_2$	$J_3$	$J_4$	$J_5$	$J_6$	$J_7$	$J_8$	$J_9$	$J_{10}$	$J_{11}$	$J_{12}$	$J_{13}$	$J_{14}$
0.00	1.00	—	—	—	—	—	—	—	—	—	—	—	—	—	—
0.25	0.98	0.12	—	—	—	—	—	—	—	—	—	—	—	—	—
0.5	0.94	0.24	0.03	—	—	—	—	—	—	—	—	—	—	—	—
1.0	0.77	0.44	0.11	0.02	—	—	—	—	—	—	—	—	—	—	—
1.5	0.51	0.56	0.23	0.06	0.01	—	—	—	—	—	—	—	—	—	—
2.0	0.22	0.58	0.35	0.13	0.03	—	—	—	—	—	—	—	—	—	—
2.4	0	0.52	0.43	0.20	0.06	0.02	—	—	—	—	—	—	—	—	—
2.5	-0.05	0.50	0.45	0.22	0.07	0.02	0.01	—	—	—	—	—	—	—	—
3.0	-0.26	0.34	0.49	0.31	0.13	0.04	0.01	—	—	—	—	—	—	—	—
4.0	-0.40	-0.07	0.36	0.43	0.28	0.13	0.05	0.02	—	—	—	—	—	—	—
5.0	-0.18	-0.33	0.05	0.36	0.39	0.26	0.13	0.05	0.02	—	—	—	—	—	—
6.0	0.15	-0.28	-0.24	0.11	0.36	0.36	0.25	0.13	0.06	0.02	—	—	—	—	—
7.0	0.30	0.00	-0.30	-0.17	0.16	0.35	0.34	0.23	0.13	0.06	0.02	—	—	—	—
8.0	0.17	0.23	-0.11	-0.29	-0.10	0.19	0.34	0.32	0.22	0.13	0.06	0.03	—	—	—
9.0	-0.09	0.25	0.14	-0.18	-0.27	-0.06	0.20	0.33	0.31	0.21	0.12	0.06	0.03	0.01	—
10.0	-0.25	0.05	0.25	0.06	-0.22	-0.23	-0.01	0.22	0.32	0.29	0.21	0.12	0.06	0.03	0.01

This is a very graphical (and I believe COMPELLING) way to look at WBFM

Basically, as  $m_f$  increases beyond 0.25 the significant contributions of  $J_1, J_2, J_3$  etc. increases.

That is, for larger “ $m_f$ ” the more important are higher order terms.

This in turn means that we need more terms in the expansion (shown in figure 7) above.

As an example, for  $m_f=1$  (radian) this means that we should consider up to and including the  $J_3$  terms.

Obviously, the larger  $m_f$ , the wider the overall “bandwidth”

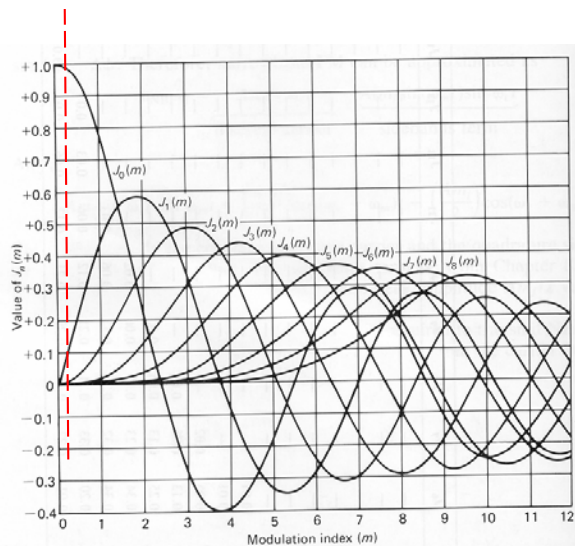


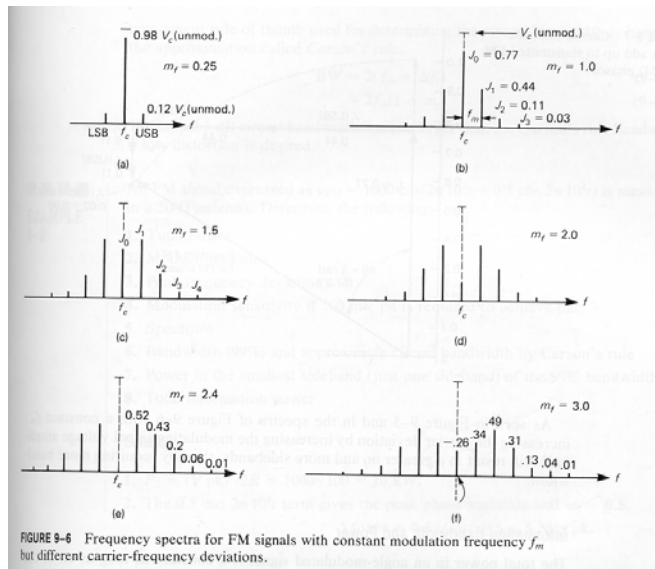
FIGURE 9-5 Amplitudes of carrier and sidebands for FM (and PM) relative to the unmodulated carrier.

This slide shows graphically how the various  $J_k$  factors are influenced by  $m_f$

Clearly, each “J” term has a regime (of  $m_f$ ) below which its contributions are negligible...

As an example, for  $m_f < 5$  we shouldn't have to worry about  $J_8$

This is in fact consistent with the TABLE of data from the previous figure.



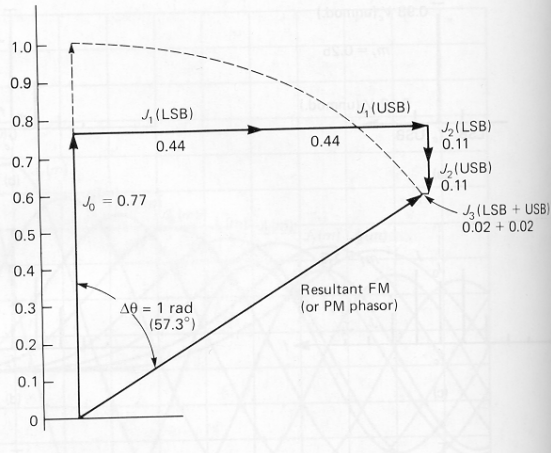
This figure shows from a spectrum point of view several examples of “J” coefficients and “ $m_f$ ” terms, up to a value of  $m_f=3$

This will be discussed further in class.

It’s basically a set of examples that are based on the previous two slides.

## WBFM for a $m_f=1$ rad

FIGURE 9-7 Carrier and sidebands add up to transmitted FM (or PM) phasor.



This final example for  $m_f=1$  (radian) Shows in a bit more detail what all the “vectors” (in a multi-frequency “vector diagram”) would look like.

It’s a VERY interesting plot and we will discuss it at some length Basically...

The length of the resultant vector is UNITY (that’s what FM means! In some sense...)

The DOMINANT terms are in  $J_0$ ,  $J_1$  and  $J_2$

The  $J_3$  term is really quite small....(look at the TABLE to see that this is a consistent result!)

The overall ANGLE DQ is called out (and  $+57.3^\circ$ )

Clearly, this is a good example and it will be discussed at some length in class.

Also, it’s fairly close to WBFM such as we will be using for SPAM!