

EE 155/255: Homework 4 solutions

1. Feedback Control

(a) Open Loop Transfer Function

Part A asks us to find the open loop transfer function of V_o to D_m , which means our transfer function $H(s) = V_o/D_m$.

We can redraw our circuit like this:

We can begin by writing the Laplace transforms on the voltages for each component:

$$\begin{aligned}\mathcal{L}\left(v_l(t) = L\frac{di}{dt}\right) & \quad V_l(s) = sLI(s) \\ \mathcal{L}\left(\frac{dv_c}{dt} = \frac{I(t)}{C}\right) & \quad sV_c(s) = \frac{I(s)}{C} \\ & \quad V_c(s) = \frac{I(s)}{sC} \\ \mathcal{L}(v_r(t) = I(t)R) & \quad V_r(s) = I(s)R\end{aligned}$$

By KVL around the loop for $D_m V_1$ and then apply the Laplace transform on it, we get

$$\begin{aligned}D_m V_1 &= v_l(t) + v_r(t) + v_c(t) \\ V_1 D_m(s) &= V_l(s) + V_r(s) + V_c(s) \\ V_1 D_m(s) &= sLI(s) + I(s)R + \frac{I(s)}{sC} \\ V_1 D_m(s) &= I(s) \left(sL + R + \frac{1}{sC} \right)\end{aligned}$$

Solving for V_o will also get us a similar equation

$$\begin{aligned}V_o &= V_r(s) + V_c(s) \\ V_o(s) &= I(s) \left(\frac{1}{sC} + R \right)\end{aligned}$$

We can divide the two equations above to solve for the open loop transfer function

$$\frac{V_o(s)}{D_m(s)} = \frac{V_1 \left(Rs + \frac{1}{LC} \right)}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Another way to approach the problem is that the transfer function from Duty Factor D_m to Output Voltage V_o is simply an impedance divider with R_{in} and C_{in} in series as the bottom resistor and L as the top one. Note that answer does not depend on current.

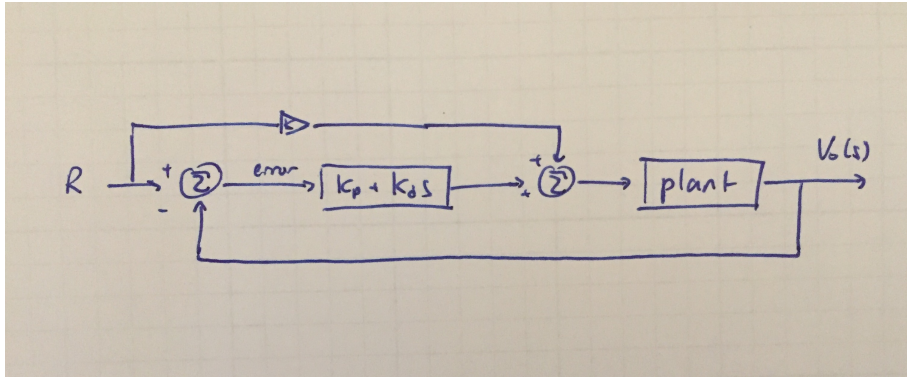
(b) Controller (13 pts)

Plugging in the appropriate values, we arrive at a transfer function for our plant P:

$$P = \frac{V_o(s)}{D_m(s)} = \frac{0.1s + 4.545e9}{s^2 + 454.5s + 4.545e8}$$

Since we would like to have zero residual error, an integral term might be nice in our controller. However, integral control has many problems and is difficult to make stable, so we will use feed forward control to tell the controller what the correct converter set-point should be.

Our controller will be:



Our feed forward path takes in a reference voltage and needs to convert it to an input for the plant. We know that during steady state of our system, in order to have 0 error, the input to our controller ($K_p + K_d s$) must be 0, since that is the error signal. This means that all control is coming from the feed forward term. What should this feed forward term be then?

$$\frac{V_o}{V_1} = D_m$$

The steady state behavior of duty factor given an output/input voltage ratio is constant. We plug in 10V for V_1 to get

$$D_m = \frac{1}{10} V_o \qquad K = \frac{1}{10}$$

For overshoot, we can estimate the time domain response of our second order system with overshoot percentage M_p based on our closed loop damping ζ .

$$M_p = 100 * e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

Without any controller, our closed loop system is simply

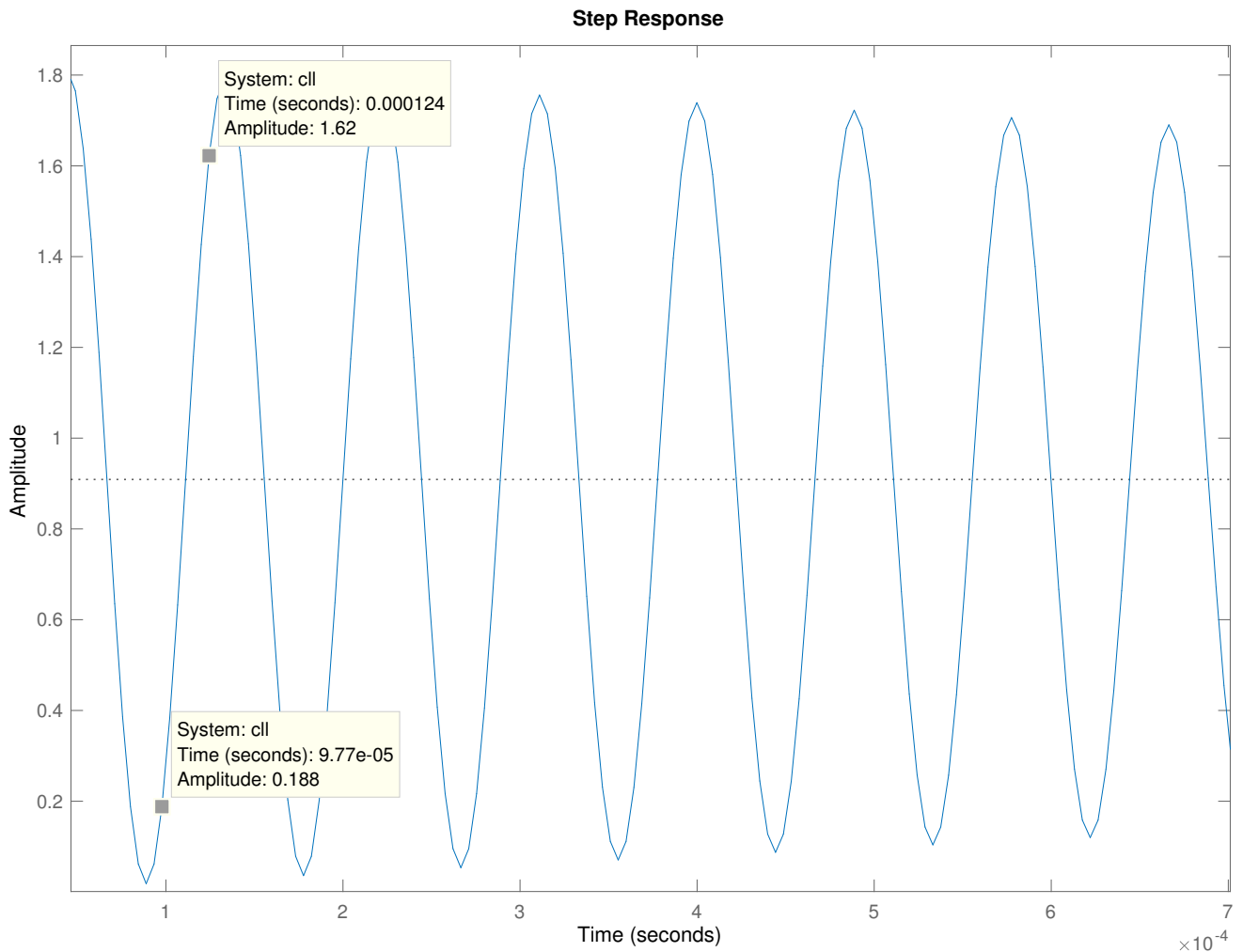
$$\frac{P}{1+P} = \frac{V_1 \left(R s + \frac{1}{LC} \right)}{s^2 + \left(\frac{R}{L} + V_1 R \right) s + \frac{1+V_1}{LC}} = \frac{0.1s + 4.545e9}{s^2 + 454.6s + 5e9}$$

Our second order system is characterized by the denominator of the closed loop equation:

$$s^2 + 454.6s + 5e9$$

Which results in a $\omega_n = \sqrt{5e9} = 7.071e5 \frac{rad}{sec}$ and damping can be calculated by $454.6 = 2\zeta\omega_n$, with a $\zeta = 0.0032$ and an overshoot of 98.99 percent. Definitely over 20 percent.

We'll add some derivative gain to ensure a better damping factor. To ensure we won't saturate the control effort of 0 to 1, we'll measure the numerical derivative of a step to our uncontrolled closed loop above first.



We see that our derivative is about $5.597e4$. Intuitively, we should choose our derivative factor K_d to be quite small since the derivative of our voltage is quite large and we want to avoid saturating our control output. If our K_d is $\frac{0.1}{5.597e4}$, we should be safe. We'll choose a K_d of $1.787e-6$. Note, 0.1 was an arbitrary choice to keep ourselves safe from saturation from multiple design iterations.

What will happen when our K_d gets too high is error will be decreasing very fast and the derivative will be negative with a large magnitude. When this derivative is multiplied by our derivative gain and then proportional control is added, our control effort will be still be negative and is impossible. From here on, we can iterate and push our controller slower and slower until we reach a physically possible duty factor with the correct overshoot. Since after choosing a K_d makes K_p solvable, we can test our system and move K_d smaller if needed.

Our controller C is now of the form $(K_p + 1.787e-6s)$.

Our new characteristic equation (denom of the transfer function) for this controller is of form

$$1 + CP$$

Solving:

$$1 + (k_p + k_d s) \left(\frac{V_1 \left(R s + \frac{1}{LC} \right)}{s^2 + \frac{R}{L} s + \frac{1}{LC}} \right)$$

$$s^2 + \left(\frac{\frac{R}{L} + k_p V_1 R + \frac{k_d V_1}{LC}}{1 + V_1 R k_d} \right) s + \left(\frac{\frac{k_p V_1}{LC} + \frac{1}{LC}}{1 + V_1 R k_d} \right)$$

$$\omega_n = \sqrt{(4.5455e9)k_p + 4.5455e8} \quad (0.1k_p + 8577.3) = 2\zeta\omega_n$$

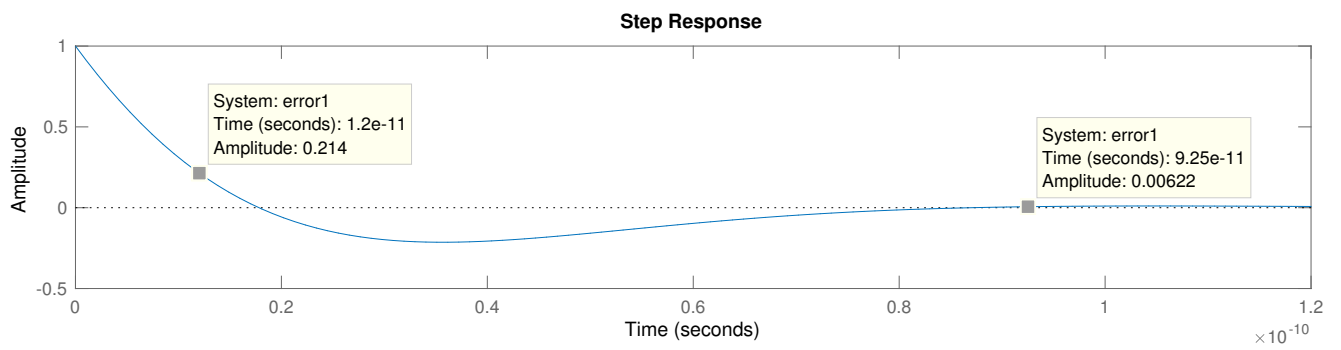
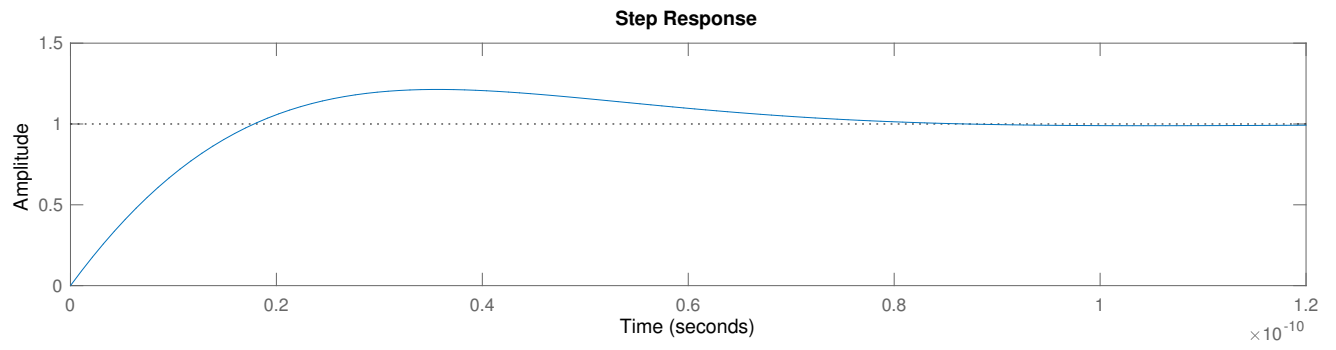
With the M_p equation above, we can back-solve for damping factor that gives us an overshoot of 10 percent. We arrive at $\zeta = 0.6901$. Solving from the above two equations and our new damping factor, we have

$$k_p = 8.6589e11$$

and a controller

$$(8.6589e11 + (1.787e-6)s)$$

Plotting the closed loop step response of the controller + feed forward $\left(\frac{CP+KP}{1+CP}\right)$
 Top graph is the output of our system and bottom is the error (inverted output).
 We can see that we see an overshoot of < 1.2 and 0 steady state error.



(c) Simulations

(d) Bode Plots

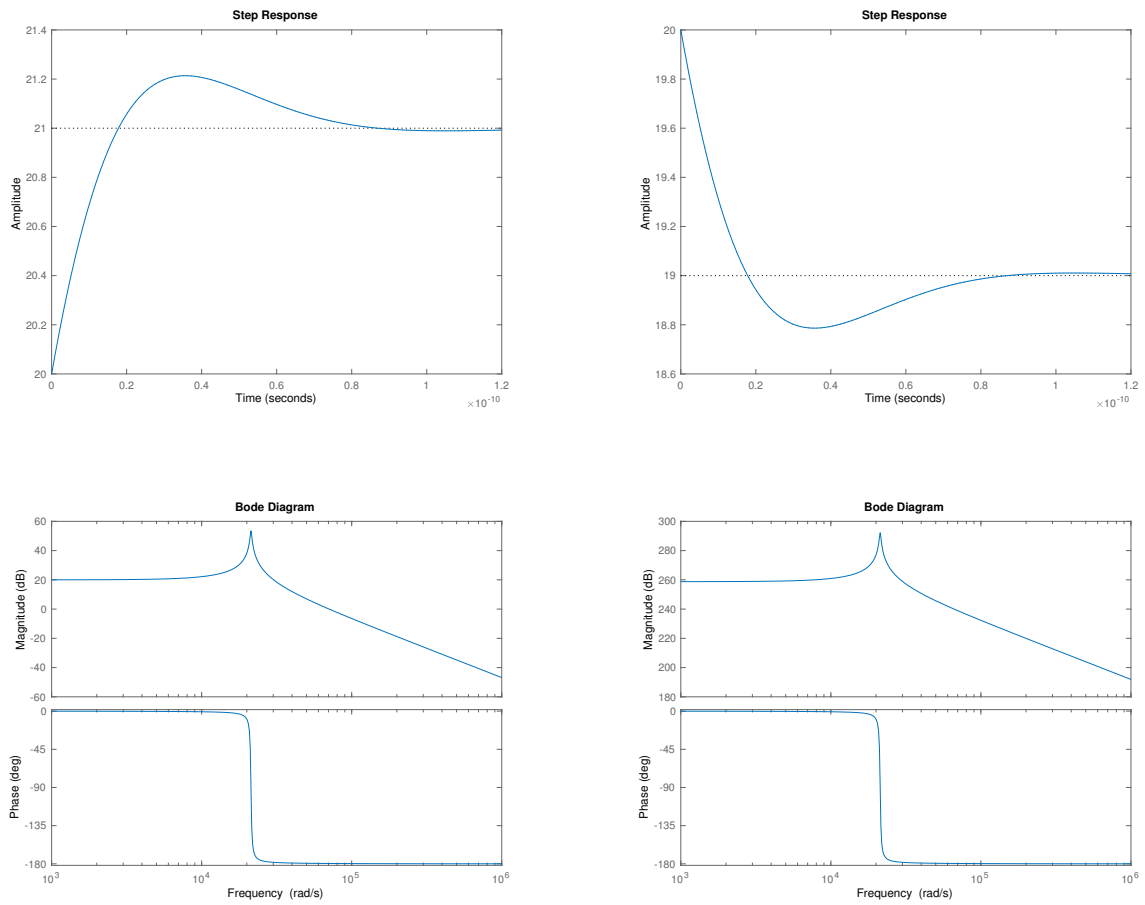


Figure 1: Left: Plant, Right: Controller and Plant

(e) Matlab

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1 clear all
2 close all
3
4 r = 0.01;
5 c = 100e-6;
6 l = 22e-6;
7 v1 = 10;
8
9 s = tf('s');
10 k = 1/10;
11
12 % Plant
13 p = v1*(r*s + 1/(l*c))/(s^2 + (r/l)*s + 1/(l*c));
14
15 % Closed loop transfer function

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16 cll = v1*(r*s + 1/(l*c))/(s^2 + (r/l + v1*r)*s + (v1+1)/(l*c));
17
18 syms kp
19 kd = 1.787e-6;
20 wn_squared = ((kp*v1)/(l*c) + 1/(l*c))/(1+v1*r*kd);
21 two_zeta_wn = (r/l + kp*v1*r + kd*v1/(l*c))/(1+v1*r*kd);
22
23 % symbolic solve for kp
24 syms zeta
25 %zeta = two_zeta_wn/(2*sqrt(wn_squared));
26 eqn = 5 == 100*exp((-zeta*pi)/(sqrt(1-zeta^2)));
27 zeta = vpa(solve(eqn, zeta),5)
28 zeta = 0.6901 % take nonnegative zeta value
29
30 eqn = 2*zeta*sqrt(wn_squared) == 0.1*kp+8677.3;
31 kp = vpa(solve(eqn, kp),5)
32 kp = 8.659e11; % second value
33
34 C = kp + kd*s;
35 G = (C*p + k*p)/(1+C*p);
36
37 subplot(2,1,1)
38 step(G);
39
40 subplot(2,1,2)
41 error1 = 1 - ((C*p + k*p)/(1+C*p));
42 step(error1);
43
44 figure
45 opt1 = stepDataOptions('InputOffset', 20, 'StepAmplitude',1);
46 step(G, opt1);
47
48 figure
49 opt2 = stepDataOptions('InputOffset', 20, 'StepAmplitude',-1);
50 step(G, opt2);
51
52 % Part D
53 figure
54 bode(p)
55
56 figure
57 bode(p*C)

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2. Motor Calculation

Single phase motor with $\alpha_i = 0$ at position 0. A 1V RMS corresponds to a peak voltage of $\sqrt{2}$, giving us a voltage equation in time of $v(t) = \sqrt{2}\sin(t)$. Torque τ is related to the motor constant K_m and current I .

- (a) If we double the angular velocity ω to 2 rad/s, we will have a voltage waveform that is both twice the frequency and twice the amplitude. Back-EMF is proportional to rotational speed. Waveform will be $v(t) = 2\sqrt{2}\sin(2t)$
- (b) In order to keep the rotor at a steady speed, we must apply a torque equal in magnitude and opposite in direction to the 2 N-m that is being applied to the shaft.

From part a), we have a motor constant $K_m = 1$, so at $\omega = 10$ rad/s, we have a back EMF of 10V RMS. Given our $K_m = 1$ we need 2A to achieve 2 N-m of torque, and so the resistance must be $10V/2A = 5 \Omega$. Power dissipated over the resistance becomes $P = I^2R = 20W$

- (c) The torque with respect to current is a sine wave (with no phase offset since we defined $\theta = 0$ as the vertical position). Since $\sin(30^\circ) = 1/2$ and $\sin(90^\circ) = 1$, the torque at 90° will be twice that at 30° , for 2 N-m total.
- (d) Since torque is proportional to current, doubling the current from part c) will double the torque from part c), for 4 N-m total.