Problem 1: Periodic Steady State Analysis

a) Since the problem states the converter is at periodic steady state, we can apply our old friend, V-s balance, top the inductor. The left voltage source is connected to the inductor with duty factor Dl. The right side is connected with duty factor Dr. Thus $V_1*D_l = V_c * D_r$. The problem asks for capacitor voltage, which we find by dividing by D_r giving $\mathbf{V_c = D_l/D_r * V_1 = 100*D_l/D_r}$. Note that using V-s balance allows us to avoid thinking about the complication of the different switching frequencies, or the impact of relative phase of the two switches.

b) Here we make the assumption that for each time step the capacitor and inductor charging/discharging are linear.
From the inductor equation dI/dt = V/L, we have ΔI = V/L* Δt. Again, using superposition, let's examine the impact of the two sides separately and add at the end.
   a. Plug in for the parameters for the Left side: $\Delta I_l = V_1/L*T_{cl}*D_l = 20*D_r$
   b. Plug in for the parameters for the Right side: $\Delta I_r = -V_c/L*T_{cl}*D_r = -0.2*V_c*D_l$
   c. Sum the two $\Delta I = T_{cl}/L * (V_1*D_l - V_c*D_r)$ $\mathbf{= 0.2 * (100*D_l - V_c*D_r)}$

For the change in capacitor voltage, we use $\Delta V_c = I_L/C * \Delta t$ where I_L is the current through the inductor and Δt is the time during which the current is allowed to flow into the capacitor by the right switch. Δt is $T_{cy} * D_l$ so $\Delta V_c = I_L/10^{-4} * 20*10^{-6} * D_l$ ➔ $\mathbf{\Delta V_c = 0.2*I_L*D_l}$

Problem 2: Motors

a) The problem is simply asking for the back emf voltage. Since back emf is proportional to K_M and ω in any motor, and the rotation gives us sin(θ), we know $V_{emf} = K_M*\omega*\sin(\theta)$. In this case, K_M=1, so $\mathbf{V_{emf} = \omega * \sin(\theta)}$

b) We will find current and voltage together. We know our driving voltage should take the form V = C*sin(θ) where C is some constant. The voltage dropped across the motor resistance is the difference between the applied voltage and the back-emf of the motor at its current speed ω. Thus by Ohm's law:

$$I = (1/R)*(V_{applied} - V_{emf})$$

$$I = (1/R) * (C*\sin(\theta) - \omega*\sin(\theta))$$

$$I = (C - \omega)/R * \sin(\theta)$$

We know Torque is $\mathcal{T} = K_m * I * \sin(\theta)$, so

$$\mathcal{T} = (C - \omega)/R * \sin^2(\theta)$$

We want to find C such that $\mathcal{T}_{avg}$ = 1N-m, so we take the integral from 0-2π and divide by 2π. Since integral of $\sin^2(\theta)$ is from 1 to 2π is π, and we can pull the constant (C- ω)/R out of the integral, we have

$$1 = ½*(C- \omega)/R$$

$$2 = (C- \omega)/R$$

$$2 = (C- \omega)/5$$

$$C- \omega = 10$$

$$C = 10+ \omega$$

Thus, our applied voltage should be **V_{app} = (10+ ω)* sin(θ)**

Plug in the value we found for C to the equation for I to find applied current

$$I = (C- \omega)/R * \sin(\theta)$$

$$I = (10+ \omega - \omega)/5 * \sin(\theta)$$

I = 2 *sin(θ)

1 Problem 3

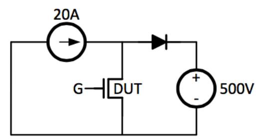


Figure 1: Boost converter under test.

Consider the boost converter shown above (figure 1) operating in the periodic steady state with a 100kHz switching frequency f_{cy}. Suppose the MOSFET has an R_{on} of 50mΩ and switches with a linear current ramp of 1A/ns for both turn-on and turn-off. Assume that the capacitance on the drain of the MOSFET is negligible and that duty factor D=0.5. Assume the diode has zero forward voltage drop but does have a reverse recovery charge of $Q_{RR} = 200nC$. Assume that the inductor and capacitor are ideal and that ripple current is negligible. Compute the switching loss and conduction loss of this converter. You may ignore turn-off losses.

(a)

Calculating the conduction loss for the converter is relatively straightforward. When the FET is on we can treat it as a resistor with resistance R_{on}. We then calculate the power dissipated as we would with a resistor ($P = I^2R$). However we note that the transistor is only on according to its duty cycle so the final equation becomes

$$P_{cond} = DI^2R_{on} = 0.5 * (20A)^2 * (50m\Omega) = \boxed{10W} \quad (1)$$

Next we have to calculate the switching loss for the FET. The problem notes that we only have to consider the **turn on** loss. First we plot the transistors voltage drop and current (figure 2).

We know the energy dissipated by the FET is non-zero when $V_{DS}I_D$ is non-zero. Immediately we can see that there is power dissipated as the current ramps up to 20A. However when the FET current reaches 20A, the diode doesn't immediately turn off because of its reverse recovery charge. This means that the FET current overshoots 20A as a function of the reverse

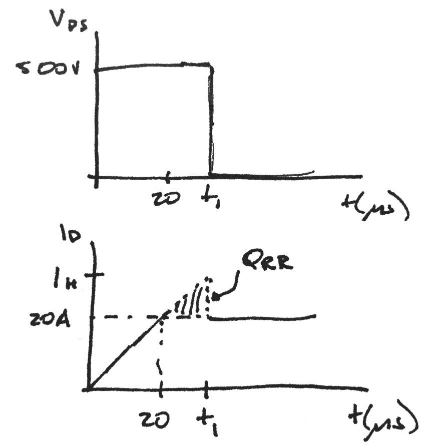


Figure 2: V_{DS} and I_D plots for FET.

recovery charge, Q_{RR}. The transistor must sink an additional Q_{RR} amount of charge with the same current ramp. This is shown in figure 2 with the shaded region. We will need to solve for the max current, I_H, and additional time spent charging, t_1. We can derive these numbers by calculating the area in the triangle above the 20A line.

$$
\begin{aligned}
Q_{RR} &= \frac{1}{2}(I_H - 20A)(t_1) = 200nC && (2)\\
&= \frac{1}{2}(1A/ns)t_1^2 && (3)
\end{aligned}
$$

$$t_1 = 20ns \tag{4}$$

$$I_H = 40A \tag{5}$$

From these numbers we can calculate the energy dissipated per switching cycle as

$$\int_{t_0}^{t_1} V_{DS} I_D dt \tag{6}$$

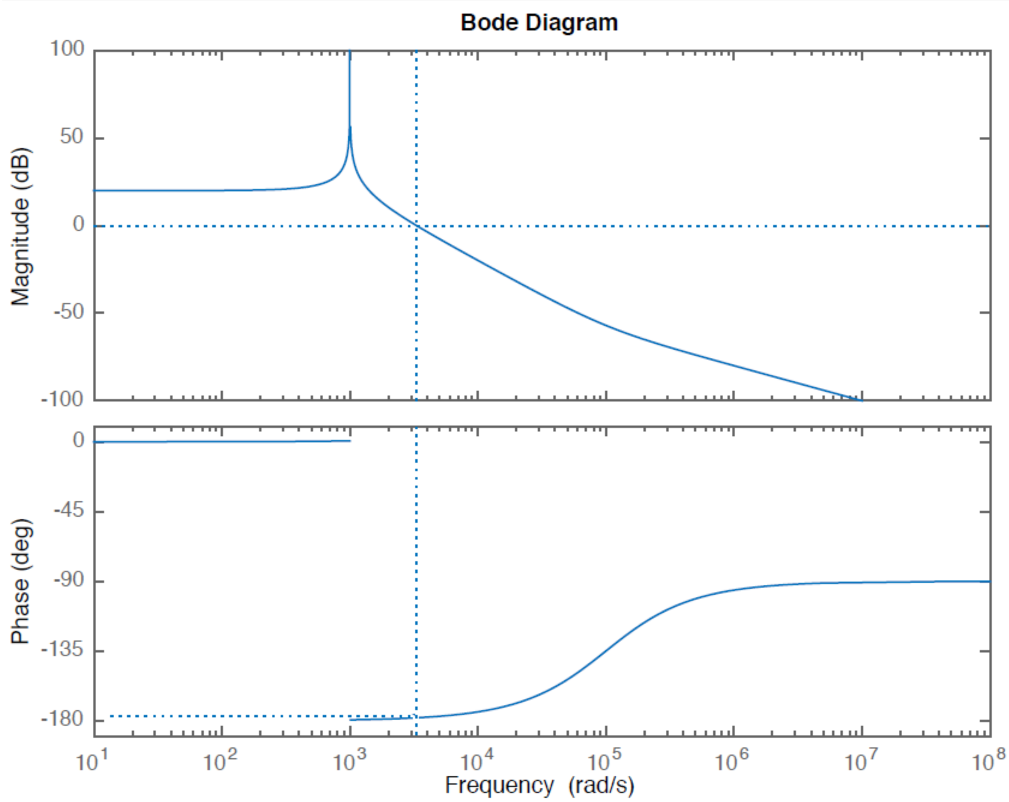
$$E_{cycle} = 500V * (\frac{1}{2} 40A * 40ns) \tag{7}$$
$$= 400\mu J \tag{8}$$

This can then be translated to the switching loss with

$$P_{sw} = f_{cy} E_{cycle} \tag{9}$$
$$= 100kHz * 400\mu J \tag{10}$$
$$= \boxed{40W} \tag{11}$$

Midterm Solutions

Problem 4: Feedback Control [20 Points]



(a) **No**, because at the unity gain frequency, phase angle $\approx -180°$ (i.e., phase margin$\approx 0°$).

(b) The open-loop frequency response, $H(s) = \frac{K(s+10^5)}{s^2+10^6}$ → $H(j\omega) = \frac{K(j\omega+10^5)}{(j\omega)^2+10^6}$

Magnitude (dB) $= 20\log_{10}(|H(j\omega)|) = 20\log_{10}\left(\left|\frac{K(j\omega+10^5)}{(j\omega)^2+10^6}\right|\right) = 20\log_{10}\left(\frac{K\left(\sqrt{\omega^2+10^{10}}\right)}{-\omega^2+10^6}\right)$

As $\omega \to 0$ rad/s, Magnitude (dB) $= 20\log_{10}\left(\frac{K\left(\sqrt{0^2+10^{10}}\right)}{-0^2+10^6}\right) = 20\log_{10}\left(\frac{K}{10}\right) = 20$ → $K = 100$

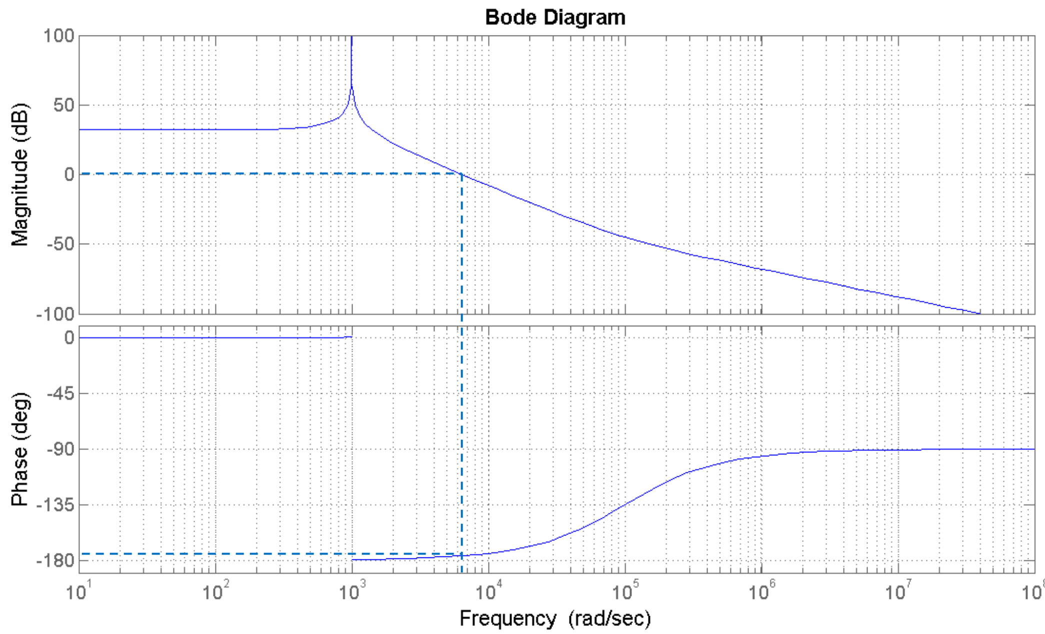
$H(s) = \frac{100(s+10^5)}{s^2+10^6}$ → $G(s) = \frac{H(s)}{1+H(s)} = \frac{100(s+10^5)}{s^2+100s+1.1\times10^7}$

Ringing will occur at $\omega_o = \sqrt{1.1\times10^7} = \mathbf{3.3166\times10^3\ rad/s}$

$\zeta = \frac{100}{2\omega_o} = \frac{100}{2(3.3166\times10^3)} \approx 0.015 \ll 1$ → This system is underdamped.

(c) The open loop system $H(s) = \frac{100(s+10^5)}{s^2+(10^3)^2}$ has **two poles at 10^3 rad/s** and **one zero at 10^5 rad/s**.

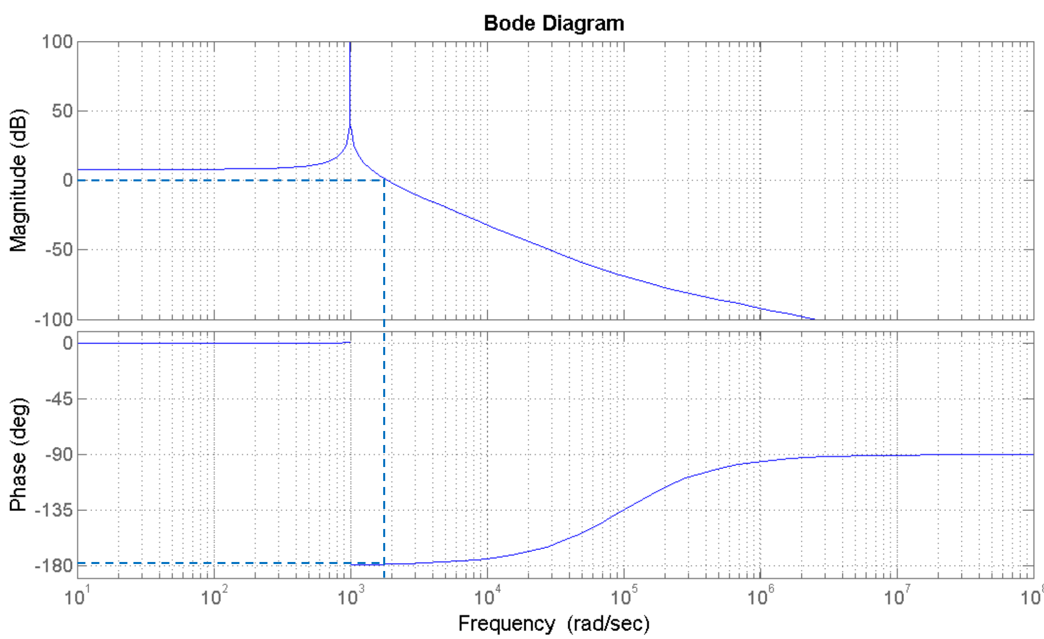
(d) (i) By increasing the DC gain by 4 (12dB), the magnitude plot shifts up by 12 dB but the phase plot remains the same. Therefore, the unity gain frequency increases and the **phase margin increases**.



$H(s) = \frac{400(s+10^5)}{s^2+10^6}$ → $G(s) = \frac{H(s)}{1+H(s)} = \frac{400(s+10^5)}{s^2+400s+4.1\times10^7}$ → $\omega_o = \sqrt{4.1\times10^7} = \mathbf{6.4\times10^3\ rad/s}$

$\zeta = \frac{400}{2\omega_o} = \frac{400}{2(6.4\times10^3)} \approx 0.031$ → **damping increases** from the original value (0.015).

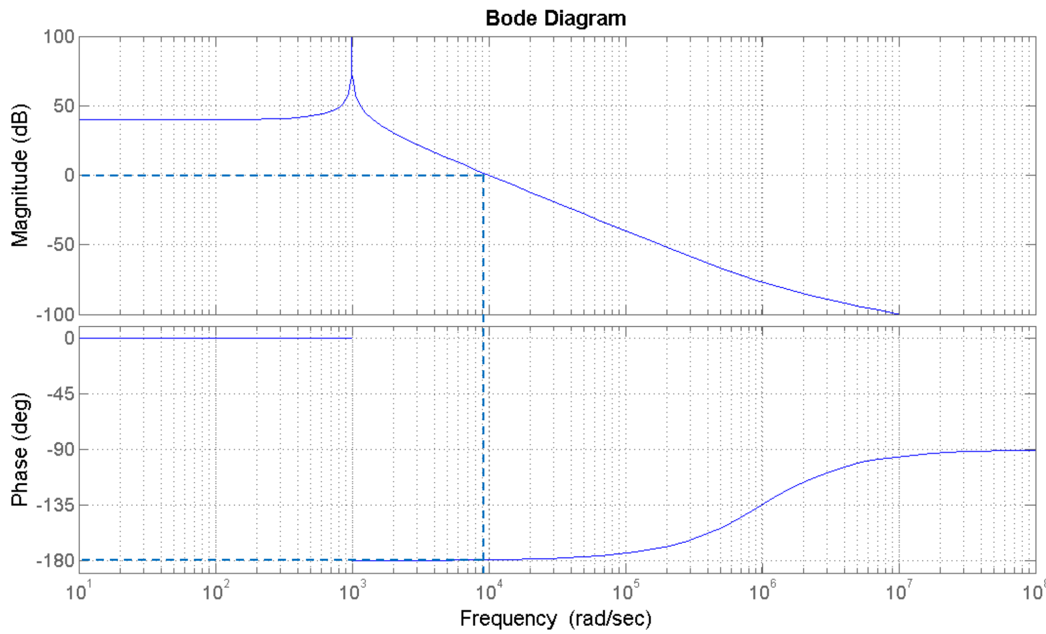
(ii) By decreasing the DC gain by 4 (12dB), the magnitude plot shifts down by 12 dB but the phase plot remains the same. Therefore, the unity gain frequency decreases and the **phase margin decreases**.



$H(s) = \frac{25(s+10^5)}{s^2+10^6} \rightarrow G(s) = \frac{H(s)}{1+H(s)} = \frac{25(s+10^5)}{s^2+25s+3.5\times10^6} \rightarrow \omega_o = \sqrt{3.5\times10^6} = \mathbf{1.87\times10^3\ rad/s}$

$\zeta = \frac{25}{2\omega_o} = \frac{25}{2(1.87\times10^3)} \approx 0.00668 \rightarrow$ **damping decreases** from the original value (0.015).

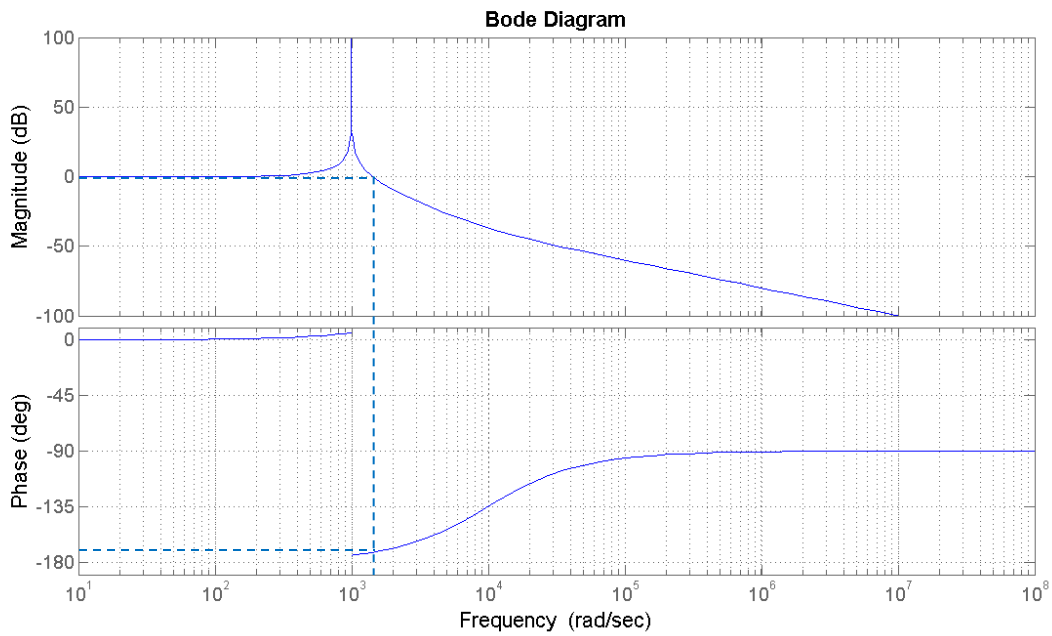
(iii) By moving the first zero up (higher frequency) by 10, the transition point from -40dB/decade slope to -20dB/decade slope in the magnitude plot moves up one decade. Similarly, the -135 degree point in the phase plot also moves up one decade but the unity gain frequency remains the same as before. Therefore, the **phase margin decreases**.



$H(s) = \frac{100(s+10^6)}{s^2+10^6} \rightarrow G(s) = \frac{H(s)}{1+H(s)} = \frac{100(s+10^6)}{s^2+100s+1.01\times10^8} \rightarrow \omega_o = \sqrt{1.01\times10^8} = \mathbf{1.005\times10^4\ rad/s}$

$\zeta = \frac{100}{2\omega_o} = \frac{100}{2(1.005\times10^4)} \approx 0.004975 \rightarrow$ **damping decreases** from the original value (0.015).

(iv) By moving the first zero down (lower frequency) by 10, the transition point in the magnitude plot from -40dB/decade slope to -20dB/decade slope moves down one decade. Similarly, the -135 degree point in the phase plot also moves down one decade but the unity gain frequency remains the same as before. Therefore, the **phase margin increases**.



$H(s) = \frac{100(s+10^4)}{s^2+10^6}$ → $G(s) = \frac{H(s)}{1+H(s)} = \frac{100(s+10^4)}{s^2+100s+2\times10^6}$ → $\omega_o = \sqrt{2\times10^6} = \mathbf{1.414\times10^3\ rad/s}$

$\zeta = \frac{100}{2\omega_o} = \frac{100}{2(1.414\times10^3)} \approx 0.035$ → **damping increases** from the original value (0.015).

Therefore, options **(i) and (iv)** will increase the phase margin, and hence the damping of the system.

Problem 5: Transformer Design [15 Points, 5Points Each]

(a) $L_M = N_1^2 A_L = (10\text{turns})^2 \times (1.4\mu\text{H/turns}^2) = \mathbf{140\mu H}$

Alternate Solution: $\mathcal{R} = \frac{l_e}{\mu_0\mu_r A_e} = \frac{(42.8\times10^{-3})}{(1530)(4\pi\times10^{-7})(31.2\times10^{-6})} = 713489.983$

$L_M = \frac{N_1^2}{\mathcal{R}} = \frac{(10)^2}{713489.983} = \mathbf{140\mu H}$

(b) $VT = N_1 B A_e = 10 \times 0.5 \times (31.2\times10^{-6}) = \mathbf{1.56\times10^{-4}}$ **Volt-Seconds**

(c) $N_1 = \frac{VT}{BA_e} = \frac{10^{-4}}{0.5\times(31.2\times10^{-6})} = 6.41$ → The primary would need to have **7 turns**.

2 Problem 6

Consider the full-bridge converter with the relevant parasitics included (Figure 3), in particular we have the transformer leakage inductance L_{leak} and switching node capacitances C_L and C_R.

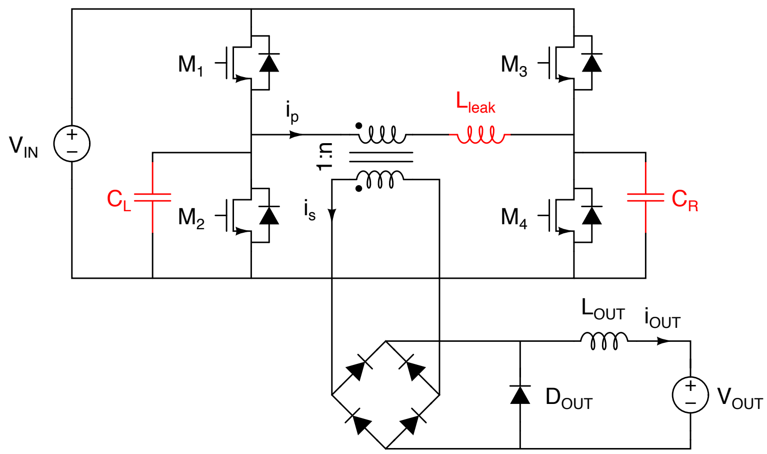


Figure 3: Full-bridge converter with key parasitics

(a)

We start out with switches M1 and M4 both on (Figure 4). This applies the full supply voltage V_{IN} across the transformer primary, and the primary and secondary currents, i_p and i_s, are both positive in the directions indicated.

Now just after we turn M1 off (Figure 5), the voltage across C_L is V_{IN} and the rectifier bridge is still conducting as before, so $i_s = i_{out}$. This secondary current is reflected across the transformer, so the primary current is $i_p = i_s/n$. Hence we see that **the output inductor L_{OUT} is the component that determines the primary current**, which is given by

$$i_p = \frac{i_{out}}{n} = \frac{20\text{ A}}{4} = \boxed{5\text{ A}}$$

(b)

Since we may assume that i_{out} is constant, i_p also remains constant, hence the time needed to fully discharge C_L and bring the switching node from V_{IN}

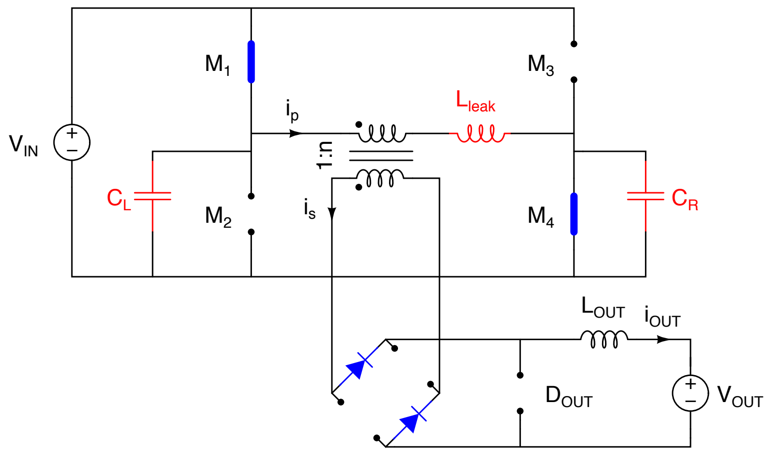


Figure 4: M1 and M4 on, start of first half of switching cycle

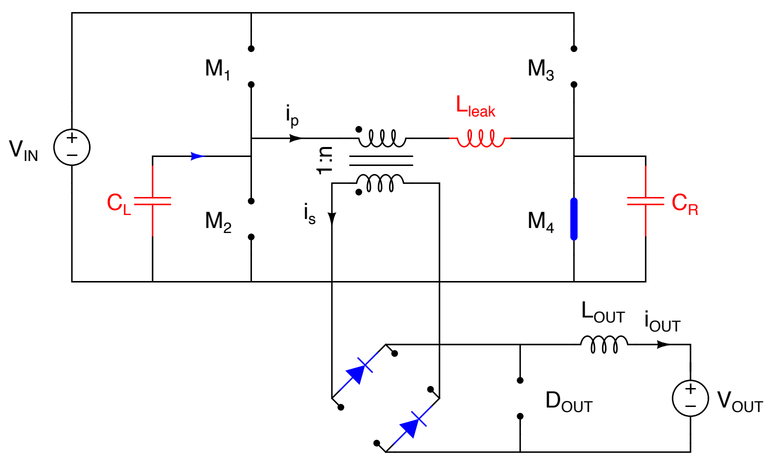


Figure 5: M1 turns off, discharging of C_L

to ground is

$$\Delta t = \frac{C_L \Delta V}{i_p} = \frac{(1\text{ nF})(400\text{ V})}{5\text{ A}} = \boxed{80\text{ ns}}$$

(c)

After C_L is completely discharged, the body diode of M2 conducts and now switch M2 can be turned on ZVS (Figure 6). This shorts the transformer primary, which is reflected across to the secondary, and the output diode

D_{OUT} begins to conduct.

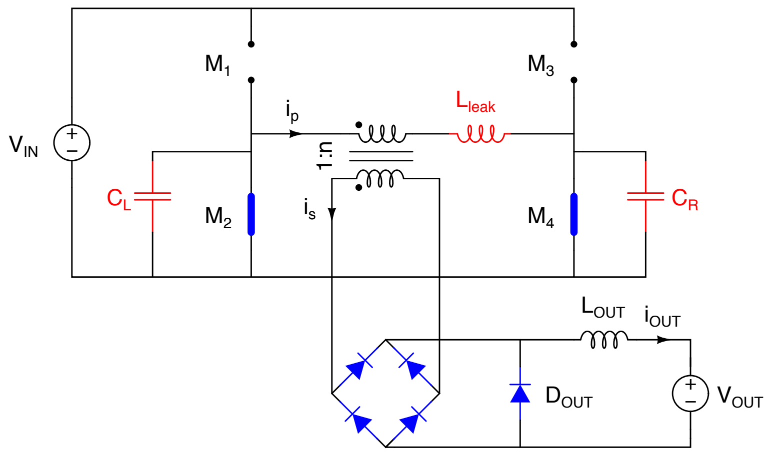


Figure 6: M2 and M4 on, current circulating

With the transformer now shorted, **the component that determines the primary current is now the leakage inductance** L_{leak}. Assuming that the current circulates through M2 and M4 losslessly, the primary current remains approximately constant at

$$i_p = \boxed{5 \text{ A}}$$

(d)

Now just after we turn M4 off (Figure 7), the voltage across C_R is 0 V and the transformer is still shorted. The energy required to charge C_R up to V_{IN} has to come from the energy stored in L_{leak}.

$$\Delta U_{req} = \frac{1}{2} C_R {V_{IN}}^2 = \frac{1}{2}(1 \text{ nF})(400 \text{ V})^2 = 80 \ \mu\text{J}$$

$$\Delta U_{stored} = \frac{1}{2} L_{leak} {i_p}^2 = \frac{1}{2}(1 \ \mu\text{H})(3 \text{ A})^2 = 12.5 \ \mu\text{J}$$

Since $\Delta U_{req} > \Delta U_{stored}$, there is not enough energy to charge C_R up to 400 V. To do so we will need to add an addition shim inductance L_{shim} at the transformer primary in series with L_{leak}.

$$\Delta U_{req} = \frac{1}{2}(L_{leak} + L_{shim}) {i_p}^2$$

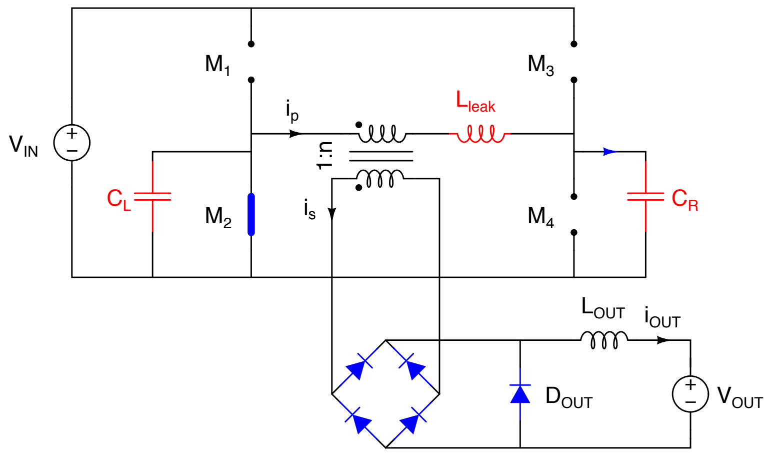


Figure 7: M4 turns off, charging of C_R

$$\implies L_{shim} = \frac{2\Delta U_{req}}{{i_p}^2} - L_{Leak} = \frac{2(80\ \mu\text{J})}{(3\ \text{A})^2} - (1\ \mu\text{H}) = \boxed{16.8\ \mu\text{H}}$$

After C_L is charged up to V_{IN}, the body diode of M3 conducts and M3 can turned on ZVS (Figure 8). The half-cycle shown in Figures 4–7 now repeats in the opposite direction.

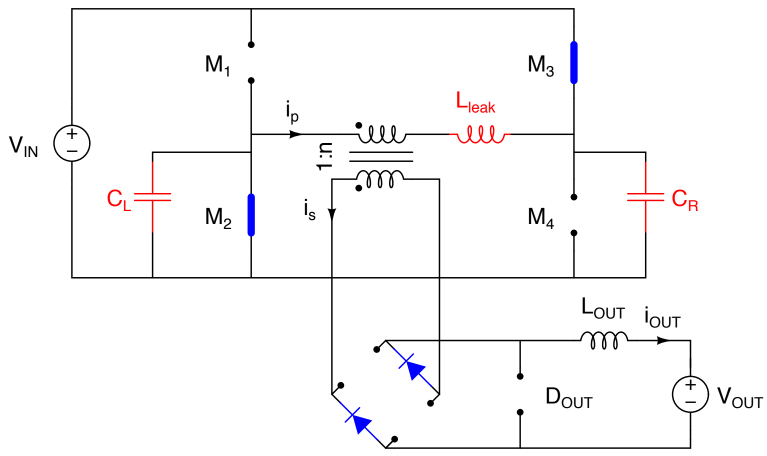


Figure 8: Life goes on, start of second half of switching cycle

3 Problem 7 (Bonus)

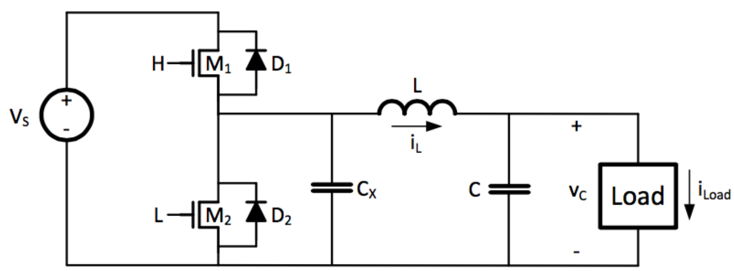


Figure 9: A quasi-square-wave converter.

Consider the quasi-square-wave (QSW) buck converter shown above where V_S=100V, V_C=40V, and i_L=10A. Suppose L=10μH and the total capacitance on the switching node including C_X is 100nF.

(a)

When the low switch (M_2) is on, what should i_L be before M_2 turns off?

In order to soft switch the converter, we need C_X to charge fully up to V_S. We can calculate what the most negative current we need in L to do so by using conservation of energy.

$$\frac{1}{2}C_X V_S^2 = \frac{1}{2}LI_L^2 \quad (12)$$

$$I_L = \sqrt{\frac{C_X}{L}}V_S \quad (13)$$

$$= 10A \quad (14)$$

However we note that this is not the current setpoint we need to switch M_2 off at. Instead we switch M_2 off earlier since the current will continue to decrease until the switching node voltage increases past V_C. First we note that the schematic when the switch is off becomes

From this we can derive the following equations

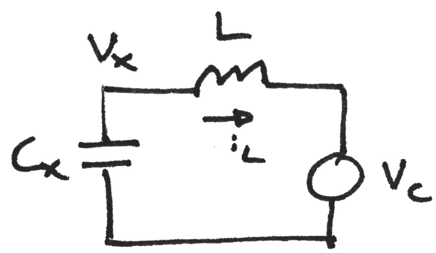


Figure 10: QSW equivalent circuit with both FETs off.

$$L\frac{dI_L}{dt} = V_X - V_{out} \tag{15}$$

$$C_X\frac{dV_X}{dt} = -I_L \tag{16}$$

From these equations we can differentiate equation 15 and arrive at the following

$$L\frac{d^2I_L}{dt^2} = \frac{dV_X}{dt} \tag{17}$$

$$L\frac{d^2I_L}{dt^2} = -\frac{1}{C_X}I_L \tag{18}$$

$$\frac{d^2I_L}{dt^2} = -\frac{1}{LC_X}I_L \tag{19}$$

We see that this is an ordinary 2nd order differential equation with the solution

$$i_L(t) = A * sin(\frac{1}{\sqrt{LC_X}}t) + B * cos(\frac{1}{\sqrt{LC_X}}t) \tag{20}$$

We then want to solve for 'B' which is the inductor current at t = 0 (when the switch should turn off). We solve for this by differentiating i_L

$$\frac{di_L}{dt}(0) = -\frac{V_C}{L} \tag{21}$$

$$= B * \frac{1}{\sqrt{LC_X}} sin(\frac{1}{\sqrt{LC_X}}(0)) \tag{22}$$

$$B = \frac{C_X}{\sqrt{L}V_C} \tag{23}$$

$$B = I_{low} \tag{24}$$

$$I_{low} = \boxed{\text{-4 A}} \tag{25}$$

(b)

If the average output current is 10A, what should i_L be when M_1 turns off?

We want the average output current to be 10A. From the previous part we know the minimum inductor current is -10A. We can then solve for the necessary max inductor current using the following

$$I_{load} = \frac{1}{2}(I_H + I_{minimum}) \tag{26}$$

$$2I_{load} - I_{\text{minimum}} = I_H \tag{27}$$

$$I_H = \boxed{30A} \tag{28}$$

This result can also be found by looking at the current waveform for the inductor current. To deliver 10A to the load, we need to set I_H sufficiently high to negate the effect of our reverse inductor current. To calculate what to set I_H to we look at the linearized current waveform through L (figure 11). We want to consider the linearized current waveform to make calculations easier since we can ignore the sinusoidal behavior near the corners. This is a good approximation since those sinusoidal behaviors are short relative to the long linear current ramps.

The approach we will take is making sure the following equation holds

$$Q_{cycle} = 10A * t_{cycle} \tag{29}$$

The first step is identifying the slopes of the current ramps

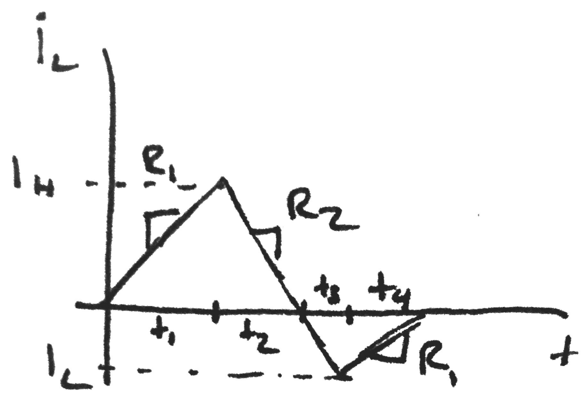


Figure 11: Linearized inductor current for a single cycle.

$$R_1 = \frac{V_S - V_C}{L} = 6A/\mu s \tag{30}$$

$$R_2 = \frac{-V_C}{L} = -4A/\mu s \tag{31}$$

From these and our previous calculated value for I_L (-10 A) we can derive the following quantities for the time periods (in μs)

$$t_1 = \frac{I_H}{R_1} = \frac{I_H}{6} \tag{32}$$

$$t_2 = \frac{I_H}{R_2} = \frac{I_H}{4} \tag{33}$$

$$t_3 = \frac{I_L}{R_2} = \frac{10}{4} \tag{34}$$

$$t_4 = \frac{I_L}{R_1} = \frac{10}{6} \tag{35}$$

We then can calculate the charge delivered in a switching cycle as

$$\frac{1}{2}\frac{I_H^2}{6} + \frac{1}{2}\frac{I_H^2}{4} - \frac{1}{2}\frac{10^2}{4} - \frac{1}{2}\frac{10^2}{6} = Q_{cycle} \tag{36}$$

Then using equation 29 we can solve for I_H

$$\frac{1}{2}\frac{I_H^2}{6} + \frac{1}{2}\frac{I_H^2}{4} - \frac{1}{2}\frac{10^2}{4} - \frac{1}{2}\frac{10^2}{6} = 10 * (\frac{I_H}{6} + \frac{I_H}{4} + \frac{10}{4} + \frac{10}{6}) \tag{37}$$

$$I_H^2(\frac{1}{12} + \frac{1}{8}) - 10I_H(\frac{1}{6}\frac{1}{4}) - (\frac{100}{8} + \frac{100}{12} + \frac{100}{4} + \frac{100}{6}) = 0 \tag{38}$$

$$0.2083I_H^2 - 4.167I_H - 62.5 = 0 \tag{39}$$

$$I_H = -10A \text{ or } 30A \tag{40}$$

$$I_H = \boxed{30A} \tag{41}$$

(c)

What will the switching frequency of this converter be with a 10A load?

Using the I_H and I_L found above we can arrive at the following

$$t_{cycle} = \frac{I_H}{6} + \frac{I_H}{4} + \frac{I_L}{4} + \frac{I_L}{6} = 16.6\mu s \tag{42}$$

Then the switching frequency is

$$f_{sw} = \frac{1}{t_{cycle}} = \frac{1}{16.6\mu s} = \boxed{60kHz} \tag{43}$$