1. Solution:

a. The sample space is
\[ \Omega = \{((i, j), (k, \ell)) : 1 \leq i, j, k, \ell \leq 6\}, \]
where \( i \) and \( j \) denote the numbers on the two dice on the first roll, and \( k, \ell \) the numbers on the two dice on the second roll. All outcomes are equally likely, so each outcome has probability \( \frac{1}{6^4} \).

If we place both quarters on 5, we win if the first roll of both dice sums to 5, and the second roll of both dice sums to 5. Let’s define three events:

\[
E = \text{on the first roll of the two dice, they sum to 5} \\
F = \text{on the second roll of the two dice, they sum to 5} \\
\text{Win} = \text{we win the game, if we put both our coins on 5}
\]

Note that \( \text{Win} = E \cap F \), so our job is to calculate \( \Pr(E \cap F) \).

Let’s calculate \( \Pr(E) \). The outcomes where \( E \) occurs are \(((1, 4), (k, \ell)), ((2, 3), (k, \ell)), ((3, 2), (k, \ell)), \) and \(((4, 1), (k, \ell))\), where \( 1 \leq k, \ell \leq 6 \). There are \( 4 \times 6^2 \) such outcomes. Therefore,
\[
\Pr(E) = \frac{4 \times 6^2}{6^4} = \frac{4}{36}.
\]

Similarly,
\[
\Pr(F) = \frac{4}{36}.
\]

Now the events \( E, F \) are independent. Therefore,
\[
\Pr(\text{Win}) = \Pr(E \cap F) = \Pr(E) \times \Pr(F) = \frac{4}{36} \times \frac{4}{36} = \frac{1}{81}.
\]

**Alternative approach:** Let’s calculate the probability that rolling two fair dice gives us a sum of 5. If we roll two dice, they sum up to 5 if we roll \((1, 4), (2, 3), (3, 2), \) or \((4, 1)\). So, there are 4 ways in which we can roll two dice so that they sum up to 5. Each of these outcomes has probability \( \frac{1}{36} \), since all 36 outcomes are equally likely.

Therefore, the probability of winning if we place both coins on 5 is:
\[
\frac{4}{36} \times \frac{4}{36} = \frac{1}{81}.
\]

b. There are two ways to place the coins if we want to maximize the probability of winning. We can either place the coins on locations 6 and 7, or on locations 7 and 8 (either gives the same chance of winning, and both are optimal). We will calculate the probability that we win by placing the coins on 6 and 7.
The sample space is as in part 1. Define the events:

\[ E_s = \text{on the first roll of the two dice, they sum to } s \]
\[ F_t = \text{on the second roll of the two dice, they sum to } t \]
\[ \text{Win}_{6,7} = \text{we win the game, if we put our coins on 6 and 7} \]

If we place the coins on locations 6 and 7, we have two ways to win:

- We win if we get a sum of 6 on the first roll of two dice and a sum of 7 on the second roll.
- We win if we get a sum of 7 on the first roll of two dice and a sum of 6 on the second roll.

These two ways are disjoint. In other words,

\[ \text{Win}_{6,7} = (E_6 \cap F_7) \cup (E_7 \cap F_6). \]

Let’s calculate the probability of each of these events. There are 6 ways to roll two dice so that they add up to 7. Thus, the probability of getting a 7 in a roll of two dice is

\[ \Pr(E_7) = \Pr(F_7) = \frac{6}{36} = \frac{1}{6}. \]

There are 5 ways to roll two dice so that they add up to 6. Therefore, the probability of getting a 6 if we roll two dice is

\[ \Pr(E_6) = \Pr(F_6) = \frac{5}{36}. \]

Moreover, the events \( E_s \) and \( F_t \) are independent for all \( s, t \). It follows that

\[ \Pr(\text{Win}_{6,7}) = \Pr((E_6 \cap F_7) \cup (E_7 \cap F_6)) \]
\[ = \Pr(E_6 \cap F_7) + \Pr(E_7 \cap F_6) \]
\[ = \Pr(E_6) \times \Pr(F_7) + \Pr(E_7) \times \Pr(F_6) \]
\[ = \frac{5}{36} \times \frac{6}{36} + \frac{6}{36} \times \frac{5}{36} \]
\[ = 2 \times \frac{6}{36} \times \frac{5}{36}. \]

The analysis for placing the coins on locations 7 and 8 is similar, and the probability of winning in that case is the same.

**Comment:** Note that if we place both coins on 7, then we need both rolls of two dice to sum up to 7. The probability of winning in this case is \( \frac{6}{36} \times \frac{6}{36} \), which is less than the probability of winning if we place the coins on locations 6 and 7 (or on 7 and 8).

What happened? It’s tempting to think that the best strategy is to place both coins on 7, since 7 is the most likely sum to get after rolling two dice. However, surprisingly, that isn’t the best possible strategy. In effect, placing the coins on locations 6 and 7 gives us almost twice as many ways to win, which more than compensates for the slight reduction in the probability of rolling a 6 (compared to the probability of rolling a 7).

2. Solution:
a. Let $A_i$ be the event that the input is $i$ and $B_i$ be the event that the output is $i$. Using the law of total probability and conditional probability we get

$$P(B_0) = P(B_0 | A_0)P(A_0) + P(B_0 | A_2)P(A_2)$$
$$= \frac{1}{2}(1 - \epsilon) + \frac{1}{4}\epsilon$$
$$= \frac{1}{2} - \frac{1}{4}\epsilon,$$

$$P(B_1) = \frac{1}{2}\epsilon + \frac{1}{4}(1 - \epsilon)$$
$$= \frac{1}{4} + \frac{1}{4}\epsilon,$$

$$P(B_2) = \frac{1}{4}\epsilon + \frac{1}{4}(1 - \epsilon)$$
$$= \frac{1}{4}.$$

As expected, the sum of the probabilities of the events $B_i$ is 1.

b. The conditional probabilities are

$$P(A_0 | B_1) = \frac{P(A_0, B_1)}{P(B_1)}$$
$$= \frac{\frac{1}{2}\epsilon}{\frac{1}{2}\epsilon + \frac{1}{4}(1 - \epsilon)}$$
$$= \frac{2\epsilon}{1 + \epsilon},$$

$$P(A_1 | B_1) = \frac{\frac{1}{4}(1 - \epsilon)}{\frac{1}{2}\epsilon + \frac{1}{4}(1 - \epsilon)}$$
$$= \frac{1 - \epsilon}{1 + \epsilon},$$

$$P(A_2 | B_1) = 0.$$

Again, note that the sum of the conditional probabilities is 1.

3. Solution:

a. The sample space of the experiment of rolling 2 five-sided dice is $\Omega = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\} = \{1, 2, 3, 4, 5\}^2$. Overall, there are 25 possible outcomes, which are all equiprobable, since the 2 dice are fair and the rolls are independent of each other. It is useful to define the following events

$A_1 =$ "at least one of the dice has a 5 showing"

$A_2 =$ "at least one of the dice has a 1 showing"
i. It is
\[
P(A) = p(\{5, 5\}) = \frac{1}{25},
\]
\[
P(A_1) = \sum_{j=1}^{5} p(\{5, j\}) + \sum_{i=1}^{4} p(\{i, 5\})
\]
\[
= \frac{5}{25} + \frac{4}{25} = \frac{9}{25}.
\]

Moreover,
\[
P(A \cap A_1) = p(\{5, 5\}) = \frac{1}{25}.
\]

We observe that
\[
P(A \cap A_1) = \frac{1}{25} \neq \frac{9}{25^2} = P(A)P(A_1).
\]

Thus, the events \(A\) and \(A_1\) are not independent.

ii. It is
\[
P(A_2) = \sum_{j=1}^{5} p(\{1, j\}) + \sum_{i=2}^{5} p(\{i, 1\}) = \frac{9}{25}.
\]

Moreover,
\[
P(A \cap A_2) = p(\emptyset) = 0 \neq \frac{9}{25^2} = P(A)P(A_2).
\]

So, the events \(A\) and \(A_2\) are not independent.

b. Let’s define the events
\(B_1 = ”I \text{ get doubles”}\)
\(B_2 = ”\text{at least one of the dice has a 3 showing”}\)

i. It is
\[
P(B) = p(\{4, 4\}) + p(\{3, 5\}) + p(\{5, 3\}) = \frac{3}{25},
\]
\[
P(B_1) = \sum_{i=1}^{5} p(\{i, i\}) = \frac{5}{25}.
\]

Moreover,
\[
P(B \cap B_1) = p(\{4, 4\}) = \frac{1}{25} \neq \frac{3 \cdot 5}{25^2} = P(B)P(B_1).
\]

Thus, the events \(B\) and \(B_1\) are not independent.

ii. It is
\[
P(B \cap B_2) = p(\{3, 5\}) + p(\{5, 3\}) = \frac{2}{25}.
\]

Thus, we have
\[
P(B_2|B) = \frac{P(B \cap B_2)}{P(B)} = \frac{2/25}{3/25} = \frac{2}{3}.
\]

iii. It is
\[
P(B \cap A_1) = p(\{3, 5\}) + p(\{5, 3\}) = \frac{2}{25}.
\]
So,
\[ P(A_1|B) = \frac{P(B \cap A_1)}{P(B)} = \frac{2/25}{3/25} = \frac{2}{3}. \]

4. Solution:

a. The sample space is
\[ \Omega = \{LLL, LLW, LWL, LWW, WLL, WLW, WWL, WWW\}, \]
where, for instance, LWL indicates that we lose the first game, win the second game, and lose the third game.

We can win in three ways: if we win all three games (WWW), if we win the first and second game and lose the third (WWL), or if we lose the first game and win the second and third game (LWW). These outcomes have probability \( qpq \), \( qp(1-q) \), and \( (1-q)pq \), respectively. Thus, the probability of winning on the ABA schedule is:
\[
Pr(\text{winning}) = qpq + qp(1-q) + (1-q)pq \\
= pq(q + 1 - q + 1 - q) \\
= pq(2-q)
\]

b. The sample space is the same as in part 1, but with a different probability assignment. We can win in the same three ways: win all three games (WWW, with probability \( pqp \)), win the first and second game and lose the third (WWL, probability \( pq(1-p) \)), lose the first and win the second and third game (LWW, probability \( (1-p)pq \)). Thus, the probability of winning on the BAB schedule is:
\[
Pr(\text{winning}) = pqp + pq(1-p) + (1-p)pq \\
= pq(p + 1 - p + 1 - p) \\
= pq(2-p)
\]

c. The BAB schedule offers us a better change of winning. The specific values of \( p,q \) don’t matter. The problem statement \( p < q \), from which it follows that \( 2-p > 2-q \). Since we know \( 0 < pq < 1 \), we can multiply both sides by \( pq \) to obtain \( pq(2-p) > pq(2-q) \). Therefore the probability of winning under the BAB schedule is always strictly larger than the probability of winning under the ABA schedule.

Comment: This might seem surprising, because the math tells us to play chess-champion Deep Blue twice and play crummy amateur Albert once, instead of the other way around. Surely it must be better to play the weaker player twice, right? Wrong.

If this seems counter-intuitive, here is what is going on. Due to the special victory condition, to win the tournament, you absolutely must win the second game; assuming you do so, you only need to win one out of the other two games. Under the ABA schedule, you have to win your one game against Deep Blue, and you get two chances to beat Albert (you only need to beat him once). Under the BAB schedule, you get two chances to beat Deep Blue (it’s enough to beat Deep Blue once, you don’t have to win both games against Deep Blue), and then you have to win your one game against Albert. Which would you rather have: two chances to beat Deep
Blue and just one chance to beat Albert, or two chances to beat Albert and just one chance to beat Deep Blue? It’s better to give yourself as many chances to beat Deep Blue as possible. The trick here is the special rule that to win the tournament you have to win two consecutive games. In comparison, under the standard rule that victory goes to whoever wins any two out of three games, then we get a more intuitive result: you are indeed better off minimizing the number of times you have to play Deep Blue and thus choosing the ABA schedule.

5. Solution:

a. For each distinguishable sequence, there are $k!$ indistinguishable ones, which one can obtain just permuting the $k$ indistinguishable objects. Since there are $n!$ possible sequences in total, the answer is $\frac{n!}{k!}$.

b. Generalizing the argument in (a), there are $\frac{n!}{k_1!k_2!...k_r!}$ distinguishable sequences.

c. Each solution can be represented in the following way

$$\cdots|\cdots|\cdots$$

where there are $n$ dots and $r-1$ bars. $x_1$ is equal to the number of dots to the left of the first bar, $x_i$ is equal to the number of dots between the $i-1$th and $i$th bar for $2 \leq i \leq r-1$ and $x_r$ is equal to the number of dots to the right of the last bar. In the example $x_1 = 3$, $x_2 = 0$, $x_3 = 2$, $x_4 = 2$, $x_5 = 4$. The number of solutions is consequently equal to the number of distinguishable sequences of bars and dots, which by the previous question is equal to

$$\frac{(n+r-1)!}{n!(r-1)!}.$$

d. Each solution can be represented in the following way

$$\cdots|\cdots|\cdots$$

where as opposed to the setting in the previous question each bar is constrained to one of $n-1$ possible positions. As a result the number of solutions is equal to the number of possible ways in which we can choose $r-1$ positions from $n-1$ possibilities: $\binom{n-1}{r-1}$.