1. Solution:
To express $\rho_{X,aY+b}$ in terms of $\rho_{X,Y}$, we proceed as follows:

$$\text{Cov}(X,aY+b) = E[(X - E(X))(aY+b - E(aY+b))]$$

$$= aE[(X - E(X))(Y - E(Y))] = a\text{Cov}(X,Y).$$

$$\text{Var}(aY+b) = a^2\text{Var}(Y).$$

Thus

$$\rho_{X,aY+b} = \frac{\text{Cov}(X,aY+b)}{\sqrt{\text{Var}(X)\text{Var}(aY+b)}} = \text{sgn}(a) \cdot \rho_{X,Y}.$$ where $\text{sgn}(a) = 1$ if $a > 0$ and $-1$ if $a < 0$.

2. Solution:

$$\text{Cov}(A,B) = E[AB] - E[A]E[B]$$

$$= E[WX + WY + X^2 + XY] - 0$$

$$= E[X^2]$$

$$= 1.$$ 

Also,

$$\text{Var}[A] = \text{Var}[X + Y]$$

$$= \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X,Y)$$

$$= 2.$$ 

Similarly, $\text{Var}[B] = 2$. Therefore,

$$\rho_{A,B} = \frac{\text{Cov}(A,B)}{\sqrt{\text{Var}[A]\text{Var}[B]}} = \frac{1}{2}.$$ 

On the other hand,

$$\text{Cov}(A,C) = E[AC] - E[A]E[C]$$

$$= E[WY + XY + WZ + XZ] - 0$$

$$= 0.$$ 

Therefore,

$$\rho_{A,C} = \frac{\text{Cov}(A,C)}{\sqrt{\text{Var}[A]\text{Var}[C]}} = 0.$$
3. Solution:

It is easy to see that

\[ E[X] = 7 \]
\[ \text{Var}[X] = \frac{(10 - 4)^2}{12} = 3 \]
\[ \text{Var}[Y] = \text{Var}[X] + \text{Var}[W] = \frac{10}{3}. \]

Also,

\[ \text{Cov}(X, Y) = E[XY] - E[X]E[Y] \]
\[ = E[X^2 + XW] - E[X]^2 \]
\[ = E[X^2] - E[X]^2 \]
\[ = \text{Var}[X] = 3. \]

Therefore, the MMSE linear estimator of \( X \) given \( Y \) is given by

\[ \hat{X} = \frac{\text{Cov}(X, Y)}{\text{Var}Y} (Y - E[Y]) + E[X] \]
\[ = \frac{9}{10} (Y - 7) + 7 \]
\[ = \frac{9}{10} Y + \frac{7}{10}. \]

The MSE of the linear estimate is given by

\[ \text{MMSE} = \text{Var}[X] - \frac{\text{Cov}(X, Y)^2}{\text{Var}Y} \]
\[ = 3 - \frac{27}{10} = \frac{3}{10}. \]

4. Solution:

We use the method of indicators. Define the event

\[ A_i = \{ \text{Data bit } i \text{ is in error} \} \text{ for } 1 \leq i \leq 10^3. \]

Let the number of errors be \( N \). Then the expected number of errors is

\[ E(N) = \sum_{i=1}^{10^3} P(A_i) = 10^3 P(A_1). \]
The probability that bit 1 is incorrectly decoded is

\[
P(A_1) = P(U_1 = 1)P\{\text{Two or more of } Y_1, Y_2, Y_3 \text{ are } 0 | U_1 = 1\} + P(U_1 = 0)P\{\text{Two or more of } Y_1, Y_2, Y_3 \text{ are } 1 | U_1 = 0\}
\]

\[
= 3 \times P\{Y_1 = 0, Y_2 = 0, Y_3 = 1 | U_1 = 1\} + P\{Y_1 = 0, Y_2 = 0, Y_3 = 0 | U_1 = 1\}
\]

\[
= 3 \times P\{Z_1 = 1, Z_2 = 1, Z_3 = 0\} + P\{Z_1 = 1, Z_2 = 1, Z_3 = 1\}
\]

\[
= 3 \times 0.009 + 0.001 = 0.028.
\]

Thus \(E(N) = 28\).

5. **Solution:**

Define the indicator random variable

\[
Y_i = \begin{cases} 
  1 & \text{if bin } i \text{ is empty} \\
  0 & \text{otherwise} 
\end{cases}
\]

Then \(X = \sum_{i=1}^{50} Y_i\) and

\[
E(X) = \sum_{i=1}^{50} E(Y_i) = 50 \times E(Y_1).
\]

But

\[
E(Y_1) = P\{\text{bin 1 is empty}\} = \prod_{i=1}^{100} P\{\text{ball } i \text{ does not fall in bin 1}\}
\]

\[
= \left( P\{\text{ball 1 does not fall in bin 1}\} \right)^{100} = \left( \frac{49}{50} \right)^{100} = 0.13262.
\]

Thus

\[
E(X) = 6.631.
\]

6. **Solution:**
a. We have

\[
E[X_N] = \sum_{i=1}^{N} E[Y_i] = \frac{N}{2}
\]

\[
E[U] = \frac{1}{2}
\]

\[
E[X_N^2] = \sum_{i=1}^{N} E[Y_i^2] + \sum_{i=1}^{N} \sum_{j \neq i} E[Y_iY_j]
\]

\[
= \sum_{i=1}^{N} E[Y_i] + \sum_{i=1}^{N} \sum_{j \neq i} E[Y_iY_j]
\]

\[
= \frac{N}{2} + \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{3}
\]

\[
= \frac{N}{2} + \frac{N(N-1)}{3}
\]

\[
= \frac{2N^2 + N}{6}
\]

\[
E[U^2] = \frac{1}{3}
\]

\[
\text{Var}[X_N] = \frac{2N^2 + N}{6} - \frac{N^2}{4} = \frac{N^2 + 2N}{12}.
\]

Also,

\[
E[X_NU] = \sum_{i=1}^{N} E[Y_iU]
\]

\[
= NE[Y_1U]
\]

\[
= N \int_{0}^{1} u^2 \, du
\]

\[
= \frac{N}{3}.
\]

Therefore,

\[
\text{Cov}(X_N, U) = \frac{N}{3} - \frac{N}{4} = \frac{N}{12}
\]

The linear MMSE estimator of \(U\) given \(X_N\) is given by

\[
\hat{U} = \frac{\text{Cov}(X_N, U)}{\sigma_{X_N}^2}(X_N - E[X_N]) + E[U]
\]

\[
= \frac{N}{12} \frac{N^2 + 2N}{12} (X_N - \frac{N}{2}) + \frac{1}{2}
\]

\[
= \frac{1}{N+2} X_N + \frac{1}{N+2}.
\]

b. clear all;clc;close all;
u = 0.84;  % Bias
N_max = 1000;  % Total sample number
samples = rand(N_max,1)<u;  % Bernoulli(u)
U_hat = zeros(1,N_max-1);

for N=2:N_max
    % count the number of head = x_N
    x_N = sum(samples(1:N));

    % Compute U_hat
    U_hat(N-1) = 1/(N+2) * x_N + 1/(N+2);
end

% plot
plot(2:N_max, U_hat,'b',2:N_max, ones(1,N_max-1)*u,'r');