1. (a) \[ P(\text{luggage reaches } N \text{ successfully}) \]
\[ = P(\text{luggage not lost at } 1, 2, \ldots, N-1) \]
\[ = \prod_{i=1}^{N-1} P(\text{luggage not lost at } i) \text{ by independence} \]
\[ = (1-p)^{N-1} \]

(b) Let \( X \) be the city where the luggage is lost
\[ P(X=i \mid \text{luggage lost before } N) \]
\[ = P(X=i \mid X \leq N-1) \]
\[ = \frac{P(X=i, X \leq N-1)}{P(X \leq N-1)} \]
\[ = \frac{P(X=i)}{1 - P(X > N-1)} \]
\[ = \frac{(1-p)^{i+1} + p}{1 - (1-p)^{N-1}} \]

Most likely to be lost at 1

(c) can pick any city, new probability of losing luggage reaching safely will be \( (1-p)^{N-2} \).
2. Grading - no double penalty
   eg. if 2(a) is incorrect
       but 2(b) is correct occ to your
       answer to 2(a) then no points
       deducted for 2(b).

(a) \( X_i \) uniformly distributed in 1-\( n \)
    \[ P(X_i = k) = \frac{1}{n} \quad \forall \ k = 1, \ldots, n \]
    \( Y_j \) - \( \text{Bin}(m, \frac{1}{n}) \)
    \[ P(Y_j = k) = \binom{m}{k} \left( \frac{1}{n} \right)^k \left( 1 - \frac{1}{n} \right)^{m-k} \]
    \( k = 0, 1, \ldots, m \)
    \( j = 1, 2, \ldots, n \)
    5 pts for \( X_i \);
    5 pts for \( Y_j \).

(b) \( Y_j \) is Binomial distributed
    \[ E(Y_j) = \frac{m}{n} \quad j = 1, \ldots, n \]
    \[ \text{Var}(Y_j) = m \frac{1}{n} \left( 1 - \frac{1}{n} \right) \quad j = 1, \ldots, n \]
    5 pts for \( E(Y_j) \);
    5 pts for \( \text{Var}(Y_j) \).

(c) (i) Let \( Z \) = no. of buckets with \( 0 \) elements
    \[ Z = \sum_{i=1}^{n} Z_i \]
    \[ Z_i = \frac{1}{0} \quad \text{if } i^{\text{th}} \text{ bucket has } 0 \text{ elem.} \]
    \[ 0 \quad \text{otherwise} \]
\[ \text{EZ} = \sum_{i=1}^{n} \text{EZ}_i; \quad \text{by linearity of expectation} \]

\[ \text{EZ}_i = P(Z_i = 1) \]
\[ = P(Y_i = 0) \]
\[ = \binom{m}{0} \left( \frac{1}{n} \right)^0 \left( 1 - \frac{1}{n} \right)^{m-0} \]
\[ = \left( \frac{1}{n} \right)^m \]

\[ \therefore \quad \text{EZ} = n \left( \frac{1}{n} \right)^m \]

(ii). \[ \text{EZ}_i \] similar to (i),

\[ \text{expectation} = nP(Y_i = 1) \]
\[ = n \binom{m}{1} \left( \frac{1}{n} \right)^1 \left( 1 - \frac{1}{n} \right)^{m-1} \]
\[ = \frac{m}{n} \left( 1 - \frac{1}{n} \right)^{m-2} \]

(iii). Similar to (i),(ii)

\[ \text{expectation} = nP(Y_i > 1) \]
\[ = n \left( 1 - P(Y_i \leq 1) \right) \]
\[ = n \left( 1 - P(Y_i = 0) + P(Y_i = 1) \right) \]
\[ = n \left( 1 - \left( \frac{1}{n} \right)^m \right) - \frac{m}{n} \left( 1 - \frac{1}{n} \right)^{m-1} \]

3 pt for (i)
3 pt for (ii)
4 pt for (iii)
(d). \( E = \{ \text{at least one bucket has } > 1 \text{ elem} \} \)
\[ = \bigcup_{i=1}^{n} \{ Y_i > 1 \} \]
\[ = \{ \text{bucket } i \text{ has } > 1 \text{ elem} \} \]
\[ P(E) = P\left( \bigcup_{i=1}^{n} E_i \right) \]
\[ \leq \sum_{i=1}^{n} P(E_i) \quad \text{[union bound]} \]
\[ = \sum_{i=1}^{n} P(Y_i > 1) \]
\[ = \sum_{i=1}^{n} \left( 1 - \left(1 - \frac{1}{n}\right)^m - \frac{m}{n} \left(1 - \frac{1}{n}\right)^{m-1} \right) \]
\[ = n \left( 1 - \left(1 - \frac{1}{n}\right)^m - \frac{m}{n} \left(1 - \frac{1}{n}\right)^{m-1} \right) \]
\[ \therefore P(E) \leq n \left[ 1 - \left(1 - \frac{1}{n}\right)^m - \frac{m}{n} \left(1 - \frac{1}{n}\right)^{m-1} \right] \]

(e). \( P(X_1 = X_2) = \sum_{i=1}^{n} P(X_1 = X_2 | X_1 = i) P(X_1 = i) \)
\[ = \sum_{i=1}^{n} \frac{1}{n} \frac{1}{n} \quad \text{[independence]} \]
\[ = \frac{1}{n} \]

\[ P(\text{at least one other elem. in same bucket as } 1) \]
\[ = 1 - P(\text{no elem. in same bucket as } 1) \]
\[ = 1 - P(X_2 \neq X_1, X_3 \neq X_1, \ldots, X_m \neq X_1) \]
\[ = 1 - \prod_{i=2}^{m} P(X_i \neq X_1) \quad \text{[independence]} \]
\[ = 1 - \left(1 - \frac{1}{n}\right)^{m-1} \]
(f). \( P(w=1) = 0 \)
& \( P(w=k) = 0 \) \( \text{for } k \geq n+1 \)

For \( 1 \leq k \leq n+1 \),

\( w = k \) if first \( k-1 \) elements go into distinct buckets & \( k^{th} \) element goes into one of those \( k-1 \) buckets

\[ P(w=k) = P(x_1, \ldots, x_{k-1} \text{ distinct} \& \text{ } x_k \text{ is one of } k-1 \text{ buckets}) \]

\[ = P(x_1, x_2, \ldots, x_{k-1} \text{ distinct}) P(x_k \text{ among } k-1 \text{ buckets}) \]

\[ = \frac{1}{n} \frac{n-1}{n} \frac{n-2}{n} \ldots \frac{n-k+2}{n} \times \frac{k-1}{n} \]

\[ \text{for } 1 \leq k \leq n+1 \]

Note: that \( P(x_1, \ldots, x_{k-1} \text{ distinct}) \) is similar to the Birthday paradox done in class.
(g) Method I:

Let $Z$ = no. of collisions

$$Z = \sum_{i=1}^{m} Z_i$$

where $Z_i = 1$ if insertion of $i^{th}$ elem. was a collision

$$\therefore \mathbb{E}Z = \sum_{i=1}^{m} \mathbb{E}Z_i = \sum_{i=1}^{m} \mathbb{P}(Z_i=1)$$

Now $\mathbb{P}(Z_i=1) = \sum_{k=1}^{n} \mathbb{P}(Z_i=1|X_i=k) \mathbb{P}(X_i=k)$

$$= \sum_{k=1}^{n} \frac{k}{n}$$

Now $\mathbb{P}(Z_i=1|X_i=k) = \mathbb{P}(\text{at least one of } X_1, \ldots, X_{i-1}\text{ is equal to } k)$

$$= 1 - \left(1 - \frac{1}{n}\right)^{i-1}$$

& $\mathbb{P}(X_i=k) = \frac{k}{n}$

$$\therefore \mathbb{P}(Z_i=1) = \sum_{k=1}^{n} \left[1 - \left(1 - \frac{1}{n}\right)^{i-1}\right] \frac{k}{n}$$

$$= n\left(1 - \left(1 - \frac{1}{n}\right)^{i-1}\right) \frac{1}{n}$$

$$\therefore \mathbb{E}Z = \sum_{i=1}^{m} \left[1 - \left(1 - \frac{1}{n}\right)^{i-1}\right]$$

$$= m - \sum_{i=1}^{m} \left(1 - \frac{1}{n}\right)^{i-1}$$

$$= m - \left[1 + \left(1 - \frac{1}{n}\right) + \ldots + \left(1 - \frac{1}{n}\right)^{m-1}\right]$$

$$= m - \left[1 - \left(1 - \frac{1}{n}\right)^{m}\right] = m - n \left[1 - \left(1 - \frac{1}{n}\right)^{m}\right]$$
(g) **Method 2.**

\[ Z = \text{no. of collisions} \]

\[ Z = \sum_{i=1}^{n} V_i \]

where \( V_i = \text{no. of collisions in } i^{th} \text{ bucket} \)

According to the definition of "collision"

\[ V_i = \begin{cases} 0 & \text{if } Y_i = 0, 1 \\ Y_i - 1 & \text{if } Y_i > 1 \end{cases} \]

\[ P(V_i = 0) = P(Y_i = 0) + P(Y_i = 1) \]

\[ P(V_i = k) = P(Y_i = k+1) \quad \text{for } k = 1, 2, \ldots m-1 \]

\[ \text{E}V_i = \sum_{k=1}^{m-1} P(V_i = k)k \]

\[ = \sum_{k=1}^{m-1} k P(Y_i = k+1) \]

\[ = \sum_{k=1}^{m-1} k \left( \frac{m}{k+1} \right) \left( \frac{1}{n} \right)^k \left( 1 - \frac{1}{n} \right)^{m-k-1} \]

\[ = \sum_{k=2}^{m} (k-1) P(Y_i = k) \]

\[ = \sum_{k=2}^{m} k P(Y_i = k) - \sum_{k=2}^{m} P(Y_i = k) \]

\[ = \left[ \frac{m}{n} - 1 \cdot \frac{1}{n} \right] \left( 1 - \frac{1}{n} \right)^{m-1} \]

\[ = \frac{m}{n} - 1 + \left( 1 - \frac{1}{n} \right)^m \]

\[ \text{E}Z = \sum_{i=1}^{n} \text{E}V_i = n \text{E}V_i = m - n \left[ 1 - \left( 1 - \frac{1}{n} \right)^m \right] \]