Solutions should be complete and concisely written. Please, mark clearly the beginning and end of each problem.

You have 2 hours. Try to solve as many problems as you can during this time, but keep in mind that you can also get a good grade by solving a subset of problems.

Points assigned to each problem are indicated in parenthesis. I recommend to look at all problems before starting.

For any clarification on the text, the TAs will be outside the room, and Andrea in Packard 272.

You can consult the Bertsekas and Tsitsiklis textbook, the reader and the lecture notes. You cannot consult other books, use computers, and in particular you cannot use the web.

Solutions should be written on the blue books. Please, write your name on each of the books.

Problem 1 (15 points)
Consider the sample space $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. In words, $\Omega$ is the set of integers between 1 and 9 (including 1 and 9). Let $\mathbb{P}$ be the uniform probability law on $\Omega$. Namely, $\mathbb{P}(\{\omega\}) = 1/9$ for each $\omega \in \Omega$.

(a) Find two events $A, B \subseteq \Omega$ that are independent, but $A, B \neq \emptyset$ and $A, B \neq \Omega$. Show that they are indeed independent.

(b) Find two events $C, D \subseteq \Omega$ that are not independent. Show that they are indeed dependent (i.e. not independent).

(c) Reconsider the events $A, B \subseteq \Omega$ that you proposed in point (a). Define a new probability law on $\Omega$, call it $\tilde{\mathbb{P}}$, for which $A, B$ are no longer independent.

Problem 2 (10 points)
Three passengers enter a plane with exactly three passenger seats. (Yes, it is a small plane!) Each of the passengers has an assigned seat. They enter one at the time, but unfortunately the first passenger to enter looses her/his boarding pass. She/he sits on a uniformly random seat.

The second passenger goes to her/his assigned seat but, if she/he finds it occupied, seats at a uniformly random seat among the remaining ones.

The third passenger seats at whatever seat is still empty.

What is the probability that the third passenger seats at her/his assigned seat?

Problem 3 (10 points)
A hat contains 100 coins, of which $n$ are fair (they land heads with probability $1/2$ and tail otherwise), and $(100 - n)$ are biased (they land heads with probability $2/3$). A coin is drawn from the hat uniformly at random and tossed twice. The first time it lands heads, and the second time it lands tails. Given this observation, what is the probability that it was one of the $n$ fair coins?
Problem 4 (20 points)

We want to store information in a solid state memory. At each position, we can store a bit, i.e. either a 0 or a 1. We read out the stored information after one year. After this period of time, each bit is corrupted independently with probability $p$. This means that, if the bit was originally $x \in \{0, 1\}$, after one year it will be $x$ with probability $1 - p$ and $1 - x$ with probability $p$.

(a) Consider a byte, i.e. a sequence of 8 consecutive bits $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$. We want to compute the probability that this byte remains unchanged. (i) Define the sample space $\Omega$ you work with; (ii) Define the probability law $P$; (iii) Define the event $G$ (for ‘good’) that no bit in the byte get changed; (iv) Compute the probability of this event.

Evaluate this probability (approximately) for $p = 0.001$ and $p = 0.005$.

(b) In order to protect the byte, we add a parity check bit, call it $c_1$ that is the modulo 2 sum of the 8 bits, namely $c_1 = b_1 \oplus b_2 \oplus \cdots \oplus b_8$. In words, $c_1$ is equal to 1 if the parity of the bits $(b_1, \ldots, b_8)$ is odd, and equal to 0 otherwise. Of course, $c_1$ is stored on the same memory and hence subject to corruptions with the same probability as the other bits. When we read the byte, we compare its parity with the parity check. If they are different, we declare that there was an error in the byte if parities do not match.

What is the probability of an undetected error? (By this we mean an error in $(b_1, \ldots, b_8)$ that passes the parity check.)

What is the probability that the parity check is not passed, conditional that there was no error in $(b_1, \ldots, b_8)$?

Evaluate these probabilities (approximately) for $p = 0.001$ and $p = 0.005$.

(c) The method at the previous point allows to detect error but not correct them. In an attempt to recover errors, we repeat each of the bits 3 times.

Consider now a single bit $b$. We now write the sequence $(b, b, b)$ in the memory. Each of these repetitions is corrupted independently as above. Assume that, at the moment of retrieving the information, we read bits $Y = (Y_1, Y_2, Y_3)$ in those positions. Assume $p < 1/2$. What is the most likely value for $b$? Call this value $\hat{b}(Y)$.

(d) (Continuing from the previous point.) We correct errors by guessing $\hat{b}(Y)$ for the value of $b$. What is the probability that we make a mistake?