Problem 1

NOTE: There can be many correct answers. The following is one of them.

(a)
\[ A = \{1, 2, 3\}, \quad B = \{3, 4, 5\}, \quad \text{with} \quad P(A) = P(B) = \frac{1}{3}, \quad P(A \cap B) = P(3) = \frac{1}{9} = P(A)P(B). \]

REMARK: It is a common mistake to define events that are not subsets of \( \Omega \). For example, consider an experiment in which we draw a number from \( \{1, 2, \ldots, 9\} \) at random twice with replacement, and let \( A \) be the event that we draw 1 in the first draw, and \( B \) be the event that we draw 2 in the second draw. However \( \Omega \) does not accommodate such experiment. A suitable sample space in this experiment is \( \{(\omega_1, \omega_2) : \omega_i \in \{1, 2, \ldots, 9\}\} \), and we also have to re-define the probability law accordingly (e.g. \( P(\{(\omega_1, \omega_2)\}) = \frac{1}{81} \)).

(b)
\[ C = \{1, 2, 3\}, \quad D = \{1, 2, 3\}, \quad \text{with} \quad P(C) = P(D) = P(C \cap D) = \frac{1}{3}. \]

(c)
\[ \tilde{P}(\{\omega\}) = \begin{cases} 
0, & \omega \in \{1, 2, 4, 5\} \\
0.6, & \omega = 3 \\
0.1, & \omega \in \{6, 7, 8, 9\} 
\end{cases} \]

It is easy to check that \( \tilde{P} \) is a valid probability law. We have \( \tilde{P}(A) = \tilde{P}(B) = \tilde{P}(A \cap B) = 0.6 \), so \( A \) and \( B \) are not independent.

Problem 2

Define \( A_i \) as the event that the \( i \)th passenger sits at the assigned seat (and \( A_i^C \) as its complement). By the law of total probability:
\[ P(A_3) = P(A_3 \cap A_1 \cap A_2) + P(A_3 \cap A_1^C \cap A_2) + P(A_3 \cap A_1 \cap A_2^C) + P(A_3 \cap A_1^C \cap A_2^C) \]

Note that if the first person sits in the correct seat, then so does the second one. That implies \( P(A_3 \cap A_1 \cap A_2^C) = 0 \). We also have:
\[ P(A_3 \cap A_1 \cap A_2) = P(A_3|A_1 \cap A_2)P(A_1 \cap A_2)P(A_1) = 1 \times 1 \times \frac{1}{3} = \frac{1}{3} \]
\[ P\left( A_1 \cap A_2^C \cap A_2 \right) = P\left( A_3 | A_2^C \cap A_2 \right) P\left( A_2 | A_2^C \right) P\left( A_2^C \right) = 0 \times P\left( A_2 | A_2^C \right) P\left( A_2^C \right) = 0 \]

in which \( P\left( A_3 | A_2^C \cap A_2 \right) = 0 \) since if the second person sits correctly and the first person does not, the first person must be the seat of the third person, and hence the third person cannot be in the correct seat.

To calculate \( P\left( A_3 \cap A_2^C \cap A_2 \right) \), note that if the first and second passengers do not sit in their assigned seats, the only case that the third person can sit correctly is when the first person sits in the second person’s seat, an event denoted by \( B \), and the second person sits in the first person’s seat, an event denoted by \( C \).

\[
P\left( A_3 \cap A_2^C \cap A_2 \right) = P\left( A_3 \cap B \cap C \right) = P\left( A_3 | B \cap C \right) P\left( C | B \right) P\left( B \right) = 1 \times 1 \times 1 = \frac{1}{6}
\]

Therefore, \( P\left( A_3 \right) = 0.5 \).

**Remark:** Formally we can define the sample space \( \Omega = \{ (\omega_1, \omega_2, \omega_3) : \omega_i \in \{1, 2, 3\}, \omega_i \neq \omega_j \} \), where \( \omega_i \) represents the person who ends up sitting at the \( i \)-th sit. Explicitly

\[
\Omega = \{ (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1) \}.
\]

Then:

\[
A_1 = \{(1, 2, 3), (1, 3, 2)\}
A_2 = \{(1, 2, 3), (3, 2, 1)\}
A_3 = \{(1, 2, 3), (2, 1, 3)\}
B = \{(2, 1, 3), (3, 1, 2)\}
C = \{(2, 1, 3), (2, 3, 1)\}
\]

\[
P\left( \{(1, 2, 3)\} \right) = \frac{1}{3} \times 1 \times 1 = \frac{1}{3}
P\left( \{(1, 3, 2)\} \right) = \frac{1}{3} \times 0 = 0
P\left( \{(2, 1, 3)\} \right) = \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{6}
P\left( \{(2, 3, 1)\} \right) = \frac{1}{3} \times 0 = 0
P\left( \{(3, 1, 2)\} \right) = \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{6}
P\left( \{(3, 2, 1)\} \right) = \frac{1}{3} \times 1 \times 1 = \frac{1}{3}
\]

As a sanity check, \( \sum_{\omega \in \Omega} P\left( \{\omega\} \right) = 1 \). Also, the probability that the third person sits correctly is

\[
P\left( \{(1, 2, 3), (2, 1, 3)\} \right) = P\left( \{(1, 2, 3)\} \right) + P\left( \{(2, 1, 3)\} \right) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}
\]

which is the same as what we obtained previously.

A common mistake in this problem is, after defining a sample space \( \Omega_1 \) (possibly different from the above \( \Omega \)), assume that \( P\left( \{\omega\} \right) = 1/|\Omega_1| \) for any \( \omega \in \Omega_1 \), i.e. a uniform distribution. This is usually not correct.
Problem 3

Let \( A \) denote the event that the drawn coin is a fair coin, \( B \) the event that the first time is head, and \( C \) the event that the second time is tail. We have

\[
P (A|B \cap C) = \frac{P (A \cap B \cap C)}{P (B \cap C)} = \frac{P (A \cap B \cap C)}{P (A \cap B \cap C) + P (A^C \cap B \cap C)}
\]

where the first equality is Bayes’ rule, and the second one is by the law of total probability. We have:

\[
P (A \cap B \cap C) = P (B \cap C|A) P (A) = P (B|A) P (C|A) P (A) = \frac{1}{2} \times \frac{1}{2} \times \frac{n}{100} = \frac{n}{400}
\]

where \( P (B \cap C|A) = P (B|A) P (C|A) \) is because the tosses are independent, conditional on \( A \) (i.e. if we already know if the coin is fair or not). Similarly,

\[
P (A^C \cap B \cap C) = P (B \cap C|A^C) P (A^C) = P (B|A^C) P (C|A^C) P (A^C) = \frac{2}{3} \times \frac{1}{3} \times \frac{100 - n}{100} = \frac{100 - n}{450}
\]

Therefore,

\[
P (A|B \cap C) = \frac{n/400}{n/400 + (100 - n)/450} = \frac{9n}{n + 800}
\]

As a sanity check, for \( n = 0 \), \( P (A|B \cap C) = 0 \), and for \( n = 100 \), \( P (A|B \cap C) = 1 \), i.e. \( P (A|B \cap C) \) is within the range from 0 to 1 for \( 0 \leq n \leq 100 \).

**REMARK:** A simple way to define the sample space is

\[
\Omega = \{(c, \omega_1, \omega_2) : c \in \{F, B\}, \omega_1, \omega_2 \in \{H, T\}\}
\]

where \( c \) represents the type of coin, and \( \omega_i \) is the result of the \( i \)-th toss. Then:

\[
A = \{(F, H, H), (F, H, T), (F, T, H), (F, T, T)\}
\]
\[
B = \{(F, H, H), (B, H, H), (F, H, T), (B, H, T)\}
\]
\[
C = \{(F, H, T), (F, T, T), (B, H, T), (B, T, T)\}
\]

Problem 4

(a)

Sample space: \( \Omega = \{\omega = (\omega_1, \omega_2, ..., \omega_8) : \omega_i \in \{0, 1\}\} \), where \( \omega_i = 0 \) if the \( i \)-th bit is not corrupted and 1 if it is corrupted.

Probability law: \( P (\{\omega\}) = p^{\sum_{i=1}^{8} \omega_i} \times (1-p)^{8-\sum_{i=1}^{8} \omega_i} \).

The event \( G = \{(0, 0, ..., 0)\} \), with \( P (G) = p^0 \times (1-p)^8 = (1-p)^8 \). For \( p \ll 1 \), \( P (G) \approx 1 - 8p \); with \( p = 0.001 \), \( P (G) \approx 0.992 \), and with \( p = 0.005 \), \( P (G) \approx 0.96 \).
To take into account the corruption of $c_1$, we define a new sample space:

$$\Omega = \{ \omega = (\omega_1, \omega_2, ..., \omega_8, \omega_9) : \omega_i \in \{0, 1\} \}$$

where $\omega_9$ corresponds to the corruption of $c_1$, and the notations share the same meaning as in part (a). The probability law is:

$$P(\{\omega\}) = p^{\sum_{i=1}^{9} \omega_i} (1 - p)^{9 - \sum_{i=1}^{9} \omega_i}$$

First we derive $P(k$ bits among 9 bits are corrupted$)$:

$$P(k \text{ bits among 9 bits are corrupted}) = \sum_{\omega \in \Omega : \sum_{i=1}^{9} \omega_i = k} \left( \sum_{i=1}^{9} \omega_i \right) P(\{\omega\}) = \sum_{\omega \in \Omega : \sum_{i=1}^{9} \omega_i = k} p^k (1 - p)^{9 - k}$$

Now note that an undetected error occurs if and only if there are an even (non-zero) number of bits among all 9 bits are corrupted, since

- if $c_1$ is corrupted, there is an error in $(b_1, ..., b_8)$ that goes undetected when there are an odd number of bits in $(b_1, ..., b_8)$ that are corrupted, so that corrupted versions of $(b_1, ..., b_8)$ and $c_1$ still satisfy the parity check,

- if $c_1$ is not corrupted, there is an error in $(b_1, ..., b_8)$ that goes undetected when there are an even (non-zero) number of bits in $(b_1, ..., b_8)$ that are corrupted, for the same reason. (There must be at least one bit corruption in $(b_1, ..., b_8)$ so that there is an error.)

Therefore,

$$P(\text{undetected error}) = P(\text{An even (non-zero) number of bits are corrupted})$$

$$= P(2 \text{ bits are corrupted}) + P(4 \text{ bits are corrupted})$$

$$+ P(6 \text{ bits are corrupted}) + P(8 \text{ bits are corrupted})$$

$$= \binom{9}{2} p^2 (1 - p)^7 + \binom{9}{4} p^4 (1 - p)^5 + \binom{9}{6} p^6 (1 - p)^3 + \binom{9}{8} p^8 (1 - p)$$

$$= 36p^2 (1 - p)^7 + 126p^4 (1 - p)^5 + 84p^6 (1 - p)^3 + 9p^8 (1 - p)$$

With $p = 0.001$, $P(\text{undetected error}) \approx 3.57 \times 10^{-5}$. With $p = 0.005$, it is $8.69 \times 10^{-4}$.

Also,

$$P(\text{parity check is not passed} \mid \text{there is no error}) = P(c_1 \text{ is corrupted} - \text{there is no error among } b_1, ..., b_8) = p$$

by independence.
(c) The question asks that, given that we observe \( Y \) and do not know \( b \), we have to guess what \( b \) is originally. (Had we known \( b \) at the time of retrieving \( Y \), there would be no point in retrieving \( Y \), or in repeating \( b \) three times as an attempt to recover it from noise!) Hence we need to find a rule \( \hat{b}(Y) \), which maps each possible value of \( Y \) to either 0 or 1. Note that \( Y \in \{000, 001, 010, 011, 100, 101, 110, 111\} \).

Let \( P_j(Y) \) denote the probability that the read bits are \( Y \) when \( b = j \). Let \( n_1(Y) \) denote the number of 1-bits in \( Y \). Then:

\[
P_1(Y) = p^{3-n_1(Y)}(1-p)^{n_1(Y)}
\]
\[
P_0(Y) = p^{n_1(Y)}(1-p)^{3-n_1(Y)}
\]

Since \( p < 1/2 \), we see that \( P_1(Y) > P_0(Y) \) if \( n_1(Y) \geq 2 \), and \( P_1(Y) < P_0(Y) \) otherwise. We thus do the following: declare \( \hat{b}(Y) = 1 \) if \( n_1(Y) \geq 2 \), and \( \hat{b}(Y) = 0 \) otherwise.

It is an exercise to verify that, if we assume \( b \) can be either 0 or 1 with equal probability (i.e. \( P(b = 0) = P(b = 1) = 0.5 \)), then the above procedure is equivalent to the following: declare \( \hat{b}(Y) = 1 \) if \( P(b = 1|Y) > P(b = 0|Y) \), and \( \hat{b}(Y) = 0 \) otherwise. This rule seems to fit the meaning of “most likely value of \( b \)” better. Since we do not know anything about \( b \), the assumption that \( P(b = 0) = P(b = 1) = 0.5 \) is a fair one to make.

(d) Let \( P_k \) (mistake) denote the probability that we make a mistake when \( b = k \).

Suppose \( b = 1 \). We make a mistake if \( n_1(Y) \in \{0, 1\} \), i.e.

\[
P_0 \text{ (mistake)} = P(n_1(Y) = 0|b = 1) + P(n_1(Y) = 1|b = 1)
= p^3 + 3p^2(1-p)
\]

Similarly, if \( b = 0 \), we make a mistake if \( n_1(Y) \in \{2, 3\} \), i.e.

\[
P_1 \text{ (mistake)} = P(n_1(Y) = 2|b = 0) + P(n_1(Y) = 3|b = 0)
= 3p^2(1-p) + p^3
\]