1.2. Sample space \( \Omega = \{ H, HT, \ldots \} \)
\[ \Omega = \{ H, TH, TTH, TTTH, \ldots \} \]
\( \Omega \) is not finite.

All outcomes cannot be equally likely because\( \Omega \) is infinite and the probabilities must sum to 1.

\[ E_n = \{ TTTT \ldots TH \} \]

\( \bigcup_{n=1}^{\infty} E_n \) denotes the event that an H occurs at some time \( n \).

Thus, \( \bigcap_{n=1}^{\infty} E_n \) denotes the event that a head never occurs.
2.3. Sample space
\[ S = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \{1, 2, 3, 4, 5, 6\}\} \]

All outcomes in \( S \) are equally likely

\[ |S| = 6^4 \]

\[ P(x_1, x_2, x_3, x_4) = \frac{1}{6^4} \text{ for } x_1, x_2, x_3, x_4 \text{ in } 1 \text{ to } 6 \]

We want to compute \( P(E) \) where \( E \) is the event that at least one 6 comes up

In this case, it is easier to compute

\[ P(E^c) = 1 - P(E) \]

\( E^c \) is the event that no 6 comes up in the 4 rolls

\[ E^c = \{(x_1, x_2, x_3, x_4) : x_1, x_2, x_3, x_4 \in \{1, 2, 3, 4, 5\}\} \]

\[ |E^c| = 5^4 \]

\[ P(E^c) = \frac{|E^c|}{|S|} = \frac{5^4}{6^4} \text{ since all outcomes are equally likely} \]

\[ P(E) = 1 - \frac{5^4}{6^4} \]
3. \( \mathcal{S} = \{ (x_A, x_B): x_A, x_B \in \{1, \ldots, 1000\} \} \)

\( x_A = \text{Alice's no.} \)
\( x_B = \text{Bob's no.} \)

All outcomes are equally likely, \( |\mathcal{S}| = 1000^2 \)

Events:

\( E_1 = \{ x_A = x_B \} \)
\( E_2 = \{ x_A > x_B \} \)
\( E_3 = \{ x_A < x_B \} \)

We are interested in \( P(E_2) \)

\[ E_1, E_2, E_3 = \mathcal{S} \]
\[ \text{and } E_1, E_2, E_3 \text{ are disjoint} \]

\[ P(E_1) + P(E_2) + P(E_3) = P(\mathcal{S}) = 1 \]

By symmetry, \( P(E_2) = P(E_3) \)

\[ P(E_2) = \frac{1 - P(E_1)}{2} \]

Now \( E_1 = \{ (x, x): x \in \{1, 2, \ldots, 1000\} \} \)

\[ |E_1| = 1000 \]

\[ P(E_1) = \frac{|E_1|}{|\mathcal{S}|} = \frac{1000}{1000^2} = \frac{1}{1000} \]

\[ P(E_2) = \frac{1 - \frac{1}{1000}}{2} = \frac{999}{2000} \]
4. \( S = \{(H, H), (H, T), (T, H), (T, T)\}\)

All outcomes are equally likely, \(1 \cdot 2 \cdot 1 = 4\)

(a) \(E_1=\) event that first toss is a head
\[E_1 = \{(H, H), (H, T)\}\]

\(E_2=\) event that both tosses are heads
\[E_2 = \{(H, H)\}\]

\[E_1 \cap E_2 = \{(H, H)\}\]

\[P(E_2|E_1) = \frac{P(E_1 \cap E_2)}{P(E_1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}\]

(b) \(E_3=\) event that at least one toss is a head.
\[E_3 = \{(H, H), (H, T), (T, H)\}\]

\[E_2 \cap E_3 = \{(H, H)\}\]

\[P(E_2|E_3) = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}\]
Random \( X = \begin{cases} 0 & \text{if Mohan knows answer} \\ 1 & \text{if Mohan guesses} \end{cases} \)

\( Y = \begin{cases} 0 & \text{if answer correct} \\ 1 & \text{if incorrect} \end{cases} \)

Given:
\[
P(Y = 0 | X = 0) = 1 \\
P(Y = 0 | X = 1) = 0.25 \\
P(X = 0) = 0.9
\]

To find: Probability that Mohan knew the answer given that the answer is correct
\[
P(X = 0 | Y = 0) = \frac{P(X = 0, Y = 0)}{P(Y = 0)}
\]

Using chain rule:
\[
P(X = 0, Y = 0) = P(X = 0)P(Y = 0 | X = 0) = 0.9 \times 1 = 0.9
\]
\[
P(X = 1, Y = 0) = P(X = 1)P(Y = 0 | X = 1) = 0.1 \times 0.25 = 0.025
\]

\[
P(Y = 0) = P(X = 0, Y = 0) + P(X = 1, Y = 0) = 0.925
\]
\[
\therefore P(X = 0 | Y = 0) = \frac{0.9}{0.925} \approx 0.973
\]