Homework #1
Due Friday, October 13

1. Signal estimation. In Lecture 3, we said that if we want to estimate a signal vector $c \mathbf{x}$ given a measurement $g$ with an unknown error $e$

$$g = c \mathbf{x} + e$$

that the value of $c$ that minimizes the error is

$$c = \frac{\langle g, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}$$

Show that this is true by starting with the error

$$||e||^2 = (g - c \mathbf{x}) \cdot (g - c \mathbf{x})$$

expand it, differentiate with respect to $c$, set it to zero, and solve for $c$.

2. Correlation coefficient. The correlation coefficient for two signals $x(t)$ and $y(t)$ is

$$\rho = \frac{\langle x, y \rangle}{||x|| \ ||y||} = \frac{1}{\sqrt{E_x E_y}} \int_{-\infty}^{\infty} x(t) y^*(t) dt$$

This is a measure of how similar the shapes of two signals are.

a. Show that $\rho$ is independent of the amplitude of either $x(t)$ or $y(t)$, and that $-1 \leq \rho \leq 1$.

b. We want to choose two waveforms for a binary communication system. One, $x(t)$, looks like

The alternatives for the other signal, $y(t)$, are

which one is the least similar (has the largest change in $\rho$).

3. Stretching impulses. It is often useful to scale impulses in time.

a. Show that

$$\delta(at) = \frac{1}{|a|} \delta(t).$$

Start by evaluating

$$\int_{-\infty}^{\infty} f(t) \delta(at) dt$$

and then argue the conclusion.
b. Show that
\[ \delta(\omega) = \frac{1}{2\pi} \delta(f) \]
where \( \omega = 2\pi f \)

c. Show that the transforms for \( \cos(\omega t) \) in \( \omega \) and \( \cos(2\pi ft) \) in \( 2\pi f \) are consistent.

4. Capacity of AWGN channel. The capacity in bits per second of an additive white Gaussian noise (AWGN) channel is
\[ C = B \log_2 \left( 1 + \frac{P}{N_0B} \right), \]
where \( P \) is the received signal power, \( B \) is the signal bandwidth, and \( N_0/2 \) is the noise power spectral density (PSD). (The total noise power is \( N_0B \).) Consider a wireless channel where received power falls off with distance \( d \) according to the formula \( P(d) = P_t(d_0/d)^3 \). Given \( d_0 = 10 \) m, transmitter power \( P_t = 1 \) W, noise PSD \( N_0 = 10^{-9} \) W/Hz, and channel bandwidth \( B = 30 \) KHz, find the capacity of this channel for transmitter-receiver distances of 100 m and 1 km.

5. Cross correlation. The cross correlation operation is used for matched filtering. If \( x(t) \) is a signal we are looking for in another signal \( y(t) \), we perform the operation
\[ (x \star y)(t) = \int_{-\infty}^{\infty} x(\tau)y^*(\tau - t)dt = \int_{-\infty}^{\infty} x(\tau + t)y^*(\tau)dt \]
Derive a simple Fourier transform theorem for the cross correlation \( \mathcal{F}\{(x \star y)(t)\} \), and the autocorrelation \( \mathcal{F}\{(x \star x)(t)\} \).