

Signals and Systems, Part 1

- ▶ A *signal* is a real (or complex) valued function of one or more real variables.
 - ▶ voltage across a resistor
 - ▶ price of Google stock at end of each trading day
 - ▶ amount of rain at 16:00 UTC as function of latitude, longitude
 - ▶ atmospheric pressure as function of location and time

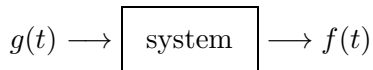
In this course the independent variable is almost always time.

- ▶ Physical signals have units, e.g., volts or psi (Si pascal = N/m²)
- ▶ Signals can (usually or in principle) be measured:
 - ▶ $g(t) \mapsto g(0)$
 - ▶ $g(t) \mapsto \int_{-\infty}^{\infty} g(u) du$ (area)
 - ▶ $g(t) \mapsto \int_{-\infty}^{\infty} |g(u)|^2 du$ (energy)

The mathematical term for a measurement is *functional*.

Signals and Systems (cont.)

- ▶ A *system* takes signals as inputs and produces signals as outputs.



- ▶ In general, the output signal depends on entirety of input signal; e.g.,
 - ▶ $f(t) = \frac{d}{dt}g(t)$
 - ▶ $f(t) = \int_{t-1}^t g(u) du$
 - ▶ $G(s) = \int_{-\infty}^{\infty} g(t)e^{-2\pi ist} dt$
- ▶ Examples of physical systems:
 - ▶ Electrical circuit: voltage in, voltage or current out
 - ▶ Building: earth shaking in, building shaking out
 - ▶ Audio amplifier

Signal Energy and Power

- ▶ The *energy* of a signal $g(t)$ is

$$\int_{-\infty}^{\infty} |g(t)|^2 dt$$

If $g(t)$ is complex valued then $|g(t)|^2$ is the square of magnitude.

- ▶ We are interested in energy only when it is finite. Common cases:
 - ▶ Bounded signal of finite duration; e.g., a pulse
 - ▶ Exponentially decaying signals (output of some linear systems with pulse input)
- ▶ Necessary condition for finite energy.
 - ▶ The energy in the “tails” of the signal must approach 0:

$$\lim_{T \rightarrow \infty} \left(\int_{-\infty}^{-T} |g(t)|^2 dt + \int_T^{\infty} |g(t)|^2 dt \right) = 0$$

- ▶ We would expect that the instantaneous power $|g(t)|^2 \rightarrow 0$, but that is *not* required. (This is only of mathematical interest.)

Signal Energy and Power (cont.)

- ▶ A signal is *periodic* if it repeats; i.e., for some $T > 0$

$$g(t + T) = g(t)$$

for every t . E.g., $\sin t$ has period 2π and $\tan t$ has period π .

- ▶ The *power* of a periodic signal $g(t)$ is the average energy:

$$P_g = \frac{1}{T} \int_a^{a+T} |g(t)|^2 dt$$

where T is the period of $g(t)$.

- ▶ The power of a general signal is a limit:

$$P_g = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt$$

This limit may be 0.

$$g_1(t) = \begin{cases} e^{-at} & t > 0 \\ 0 & t < 0 \end{cases} \quad \text{or} \quad g_2(t) = \frac{1}{1 + |t|}$$

Units of Power

- ▶ If a signal $g(t)$ measured in volts is applied to a load resistor R , then the power in watts is

$$P = \frac{g(t)^2}{R}$$

Normally we do not care about the load, so we normalize to $R = 1$.

- ▶ In many applications, the effect of the signal varies as the log of the signal; e.g., human hearing and sight.
- ▶ Power can be expressed in *decibels (dB)*. dB are logarithmic and relative to some standard power.

If P is measured in watts, then

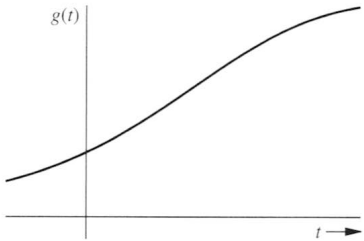
- ▶ power in dBW is $10 \log_{10} P$ (power relative to 1 W)
- ▶ power in dBm (or dBmW) is $\log_{10}(1000 P) = 30 + 10 \log_{10} P$
- ▶ One bel (B) is too large to be useful.

The bel is named for Alexander Graham Bell (1847–1922). The dB was adopted by NBS in 1931. It is *not* an SI unit.

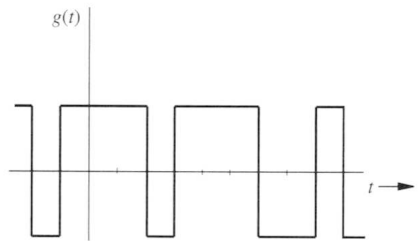
Classification of Signals

- ▶ Signals have a variety of characteristics.
 - ▶ values can be continuous or discrete
 - ▶ time variable can be continuous or discrete
 - ▶ deterministic or random
- ▶ For deterministic signals:
 - ▶ continuous time, continuous valued (mathematics)
 - ▶ continuous time, discrete valued
 - ▶ discrete time, continuous valued (digital signal processing)
 - ▶ discrete time, discrete valued (digital switching)
- ▶ Time can be restricted to a finite interval
 - ▶ Periodic signals are determined by values on a finite interval.

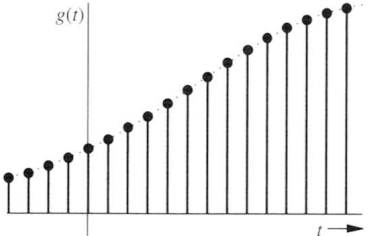
Classification of Signals (cont.)



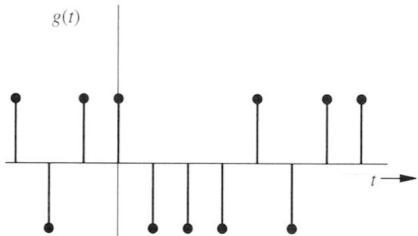
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(b)



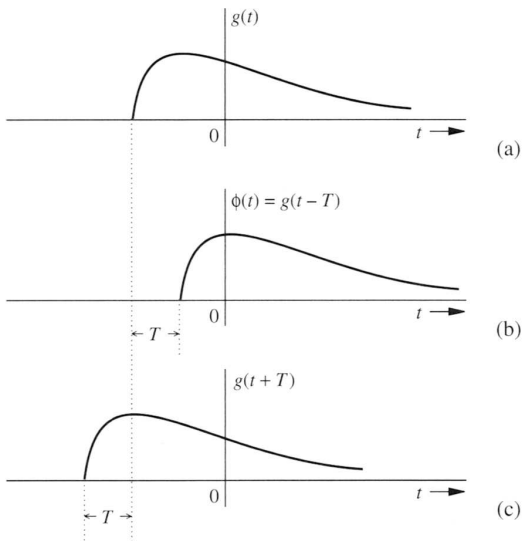
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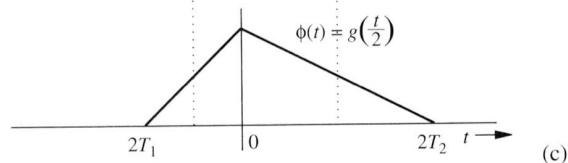
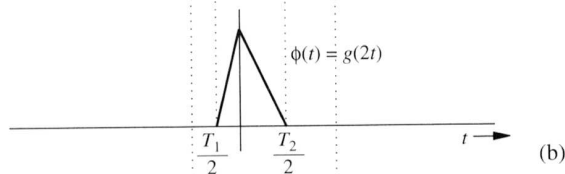
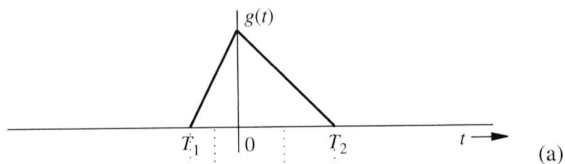
Operations on Signals

- ▶ Time shifting/delay: $g(t \pm T)$



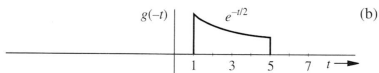
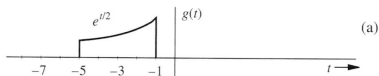
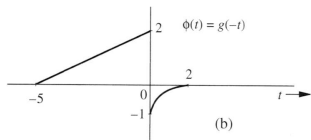
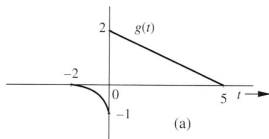
Operations on Signals (cont.)

- ▶ Time scaling: $g(at)$ stretches ($0 < a < 1$) or squeezes ($a > 1$)



Operations on Signals (cont.)

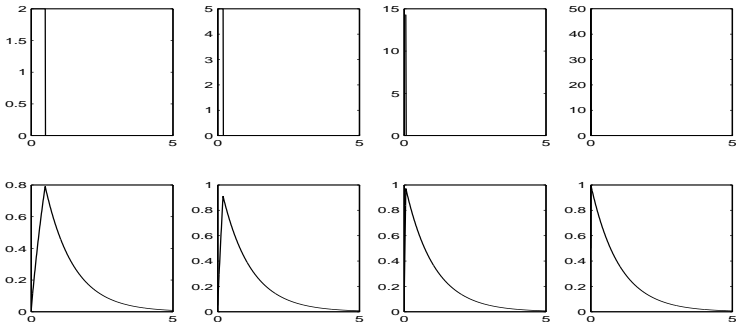
- ▶ Time reversal: $g(-t)$



- ▶ Each of these operations corresponds to a *linear* system.

Unit Impulse Signal

- ▶ Most physical systems have same output for any narrow pulse with a given area. E.g., RC circuit:



- ▶ The *unit impulse* signal is abstraction of infinitely narrow signal with area 1. Paul A. M. Dirac “defined” $\delta(t)$ by

$$\delta(t) \neq 0 \text{ if } t \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1$$

Warning: the energy of $\delta(t)$ is not defined.

Properties of Unit Impulse Signal

- ▶ Sampling property:

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi(t)\delta(t - T) dt &= \int_{-\infty}^{\infty} \varphi(t + T)\delta(t) dt \\ &= \int_{-\infty}^{\infty} \varphi(T)\delta(t) dt = \varphi(T) \int_{-\infty}^{\infty} \delta(t) dt = \varphi(T)\end{aligned}$$

In more rigorous mathematics, the sampling property *defines* the unit impulse as a *generalized function*.

- ▶ Convolution:

$$\varphi(t) * \delta(t) = (\varphi * \delta)(t) = \int_{-\infty}^{\infty} \varphi(\tau)\delta(t - \tau) d\tau = \varphi(t)$$

- ▶ Multiplication by a function:

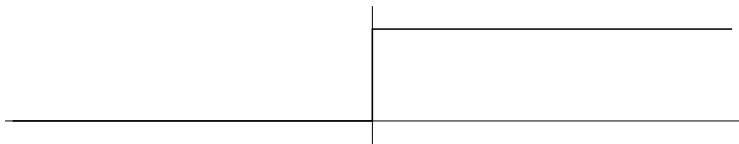
$$\varphi(t)\delta(t) = \varphi(0)\delta(t)$$

- ▶ Fourier transform of unit impulse is constant 1 at all frequencies.

Unit Step Function $u(t)$

- ▶ The Heaviside unit step function is defined by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$



The unit step function corresponds to turning on at time 0.

- ▶ Unit step is integral of unit impulse:

$$u(t) = \int_{-\infty}^t \delta(u) du \Rightarrow \delta'(t) = u(t)$$

Oliver Heaviside (1850-1925) was a self-taught English electrical engineer, mathematician, and physicist who adapted complex numbers to the study of electrical circuits, invented mathematical techniques to the solution of differential equations (later found to be equivalent to Laplace transforms), reformulated Maxwell's field equations in terms of electric and magnetic forces and energy flux, and independently co-formulated vector analysis.

The sinc function

- ▶ In EE 102A we used the definition

$$\text{sinc}_\pi(t) = \frac{\sin \pi t}{\pi t}.$$

This function is 1 at $t = 0$ and zero at the nonzero integers.

We call this sinc_π because it includes factor π factor in its argument.

- ▶ In this course, we define sinc by

$$\text{sinc}(t) = \frac{\sin(t)}{t}.$$

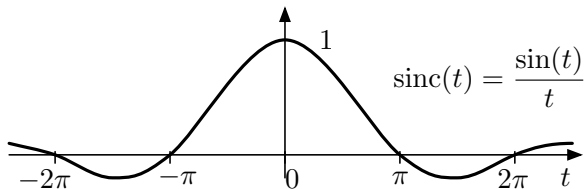
This also has an amplitude of 1 at $t = 0$ but has zeros at multiples of π .

- ▶ The two are related by

$$\text{sinc}(\pi t) = \text{sinc}_\pi(t).$$

The sinc function

Graph of $\text{sinc}(t)$.



Periodic Signals

- ▶ A signal g is called *periodic* if it repeats in time; i.e., for some $T > 0$,

$$g(t + T) = g(t)$$

for all t .

- ▶ If g is periodic, its period is the smallest such T .
- ▶ Examples: trigonometric functions are periodic.
 - ▶ Period of $\cos t$ is 2π
 - ▶ Period of $\tan t$ is π .
 - ▶ Period of square wave $\text{sgn}(\sin(2\pi t))$ is 1
- ▶ Stretching: the period of $g(mT)$ is T/m
- ▶ If g and f are periodic, their common period is $\text{LCM}(T_g, T_f)$. E.g., period of $\sin \pi t + \sin(2\pi t/5)$ is $\text{LCM}(2, 5) = 10$.

Periods must be *commensurable*, integer multiples of some shorter time.

Fourier Series

- ▶ Periodic signals can be written as the sum of sinusoids whose frequencies are integer multiples of the *fundamental frequency* $f_0 = 1/T_0$.
- ▶ The most general representation uses complex exponential functions.

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 n t}$$

Note the factor of 2π since we are using frequency in cycles/second (Hz).

- ▶ In general, Fourier series coefficients C_n are complex numbers even when the signal is real valued.
- ▶ The Fourier series coefficients can be computed by

$$C_n = \frac{1}{T_0} \int_a^{a+T_0} g(t) e^{-j2\pi f_0 n t} dt$$

The integral is over any period of the signal.

Fourier Series Examples

- ▶ Sinesuoids have a finite number of terms. By Euler's formula,

$$e^{it} = \cos t + i \sin t$$

Therefore

$$\cos t = \frac{e^{it} + e^{-it}}{2} \quad \text{and} \quad \sin t = \frac{e^{it} - e^{-it}}{2i}$$

The Fourier series coefficients for $\cos t$ are, $\dots C_{-2}, C_{-1}, C_0, C_1, C_2, \dots$

$$\dots, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, \dots$$

and for $\sin t$ are

$$\dots, 0, 0, -\frac{1}{2i} = \frac{i}{2}, 0, \frac{1}{2i} = -\frac{i}{2}, 0, 0, \dots$$

Fourier Series of Square Wave

- ▶ Square wave with period 2π defined over interval $[-\pi, \pi]$ by

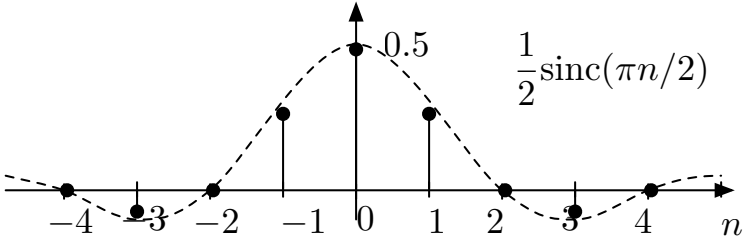
$$w(t) = \begin{cases} 1 & |t| < \pi/2 \\ 0 & -\pi < |t| < \pi/2 \end{cases}$$

Fourier series coefficients: if $n > 0$,

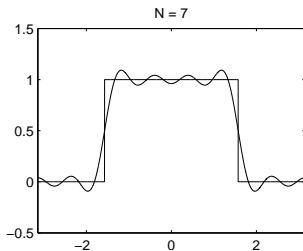
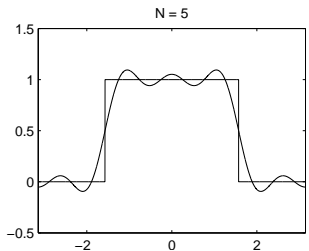
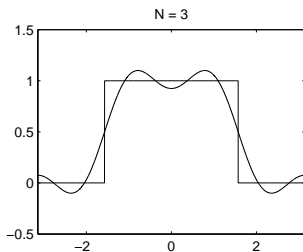
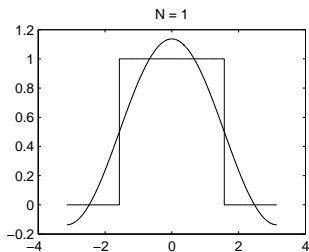
$$\begin{aligned} C_n &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} 1 \cdot e^{-jnt} dt \\ &= \frac{1}{2\pi} \left. \frac{e^{-jnt}}{-jn} \right|_{-\pi/2}^{\pi/2} \\ &= \frac{1}{2\pi} \frac{e^{\pi jn/2} - e^{-\pi jn/2}}{jn} \\ &= \frac{1}{2} \frac{\sin \pi n/2}{\pi n/2} = \frac{1}{2} \operatorname{sinc}(\pi n/2) = \frac{1}{\pi n} \quad (n \text{ odd}) \end{aligned}$$

$C_0 = \frac{1}{2}$, the average value of the square wave.

Fourier



Square Wave (cont.)



The overshoot is an example of the Gibbs' phenomenon.

The overshoot is $\approx 9\%$ and occurs no matter how many terms are used.

Fourier Series Alternative Forms

- ▶ Euler's formula $e^{j\theta} = \cos \theta + j \sin \theta$ allows us to represent periodic signals as sums of sines and cosines:

$$g(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T_0}nt\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T_0}nt\right)$$

The coefficients are

$$a_n = \frac{2}{T_0} \int_0^{T_0} g(t) \cos(2\pi f_0 nt) dt$$

$$b_n = \frac{2}{T_0} \int_0^{T_0} g(t) \sin(2\pi f_0 nt) dt$$

- ▶ A third compact form combines the $\sin(\cdot)$ and $\cos(\cdot)$ terms into phase shifted $\cos(\cdot)$ terms

$$g(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n2\pi f_0 t + \Theta_n)$$

Each frequency component is described by amplitude and phase.

Types of Systems

For theoretical and practical reasons, we restrict attention to systems and have useful properties and represent the physical world.

- ▶ Causal
- ▶ Continuous
- ▶ Stable
- ▶ Linear
- ▶ Time invariant

Fundamental fact: every linear, time-invariant system (LTIS) is characterized by

- ▶ Impulse response: $w(t) = h(t) * v(t)$
- ▶ Transform function: in frequency domain, $W(f) = H(f) \cdot V(f)$

Signals as Vectors

A signal $g(t)$ defined for a finite number of time variables,

$$g(t_k) = g_k, \quad k = 1, \dots, n$$

can be considered to be a vector of dimension n :

$$\mathbf{g} = (g_1, g_2, \dots, g_n)$$

The Euclidean *norm* (or size or magnitude) is

$$\|\mathbf{g}\| = \sqrt{|g_1|^2 + \dots + |g_n|^2} = \left(\sum_{k=1}^n |g_k|^2 \right)^{\frac{1}{2}}$$

(This definition works for complex-valued signals.)

The norm is used to measure how far apart are two signals, e.g., to tell how good an estimate is:

$$\text{square error} = \|\hat{\mathbf{g}} - \mathbf{g}\|^2 = \left(\sum_{k=1}^n (|\hat{g}_k - g_k|^2) \right)$$

Orthogonality

If two signals are orthogonal, then by Pythagoras' theorem,

$$\begin{aligned}\|f\|^2 + \|g\|^2 &= \|f + g\|^2 \\ &= (f + g) \cdot (f + g) \\ &= f \cdot f + f \cdot g + g \cdot f + g \cdot g = \|f\|^2 + \|g\|^2 + 2(f \cdot g)\end{aligned}$$

Thus two signals are orthogonal if and only inner product $f \cdot g = 0$.
(In this case the energy of the sum is the sum of the energies.)

Recall that $f \cdot g = \|f\| \|g\| \cos \theta$, where θ is the angle between f and g .

- ▶ $\theta = 0 \Rightarrow f$ and g point in exactly the same direction
- ▶ $0 < \theta < \pi/2 \Rightarrow f$ and g point in the same general direction
- ▶ $\theta = -\pi/2 \Rightarrow f$ and g are perpendicular
- ▶ $\pi/2 < \theta < \pi \Rightarrow f$ and g point in opposite general direction
- ▶ $\theta = -\pi \Rightarrow f$ and g point in exactly the opposite direction

Signals as Vectors, II

For most signal processing applications, the number of samples is much larger than 3, so we cannot easily visualize signals as vectors.

In fact, n might be infinite, e.g., $t_k = k\Delta t$ for $k = 0, 1, 2, \dots$ or even $-\infty < k < \infty$.

When the dimension is infinite, the signal's norm may be infinite. If

$$\|\mathbf{g}\|^2 = \sum_k |g_k|^2 < \infty$$

the signal has finite energy and is said to belong to L^2 .

The energy of the sampled signal depends on the sampling interval Δt . The power in each interval is $\|g_k\|^2$, so total energy is

$$\sum_k |g_k|^2 \Delta t$$

Gauss determined the orbit of Ceres using only 24 samples and an early version of the FFT.

Signals as Vectors, III

If the sample interval goes to zero, we obtain a continuous-time signal with energy

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \lim_{\Delta t \rightarrow 0} \sum_k |g_k|^2 \Delta t$$

The class of finite energy signals is named L^2 or $L^2(-\infty, +\infty)$.

The two major tasks of signal processing are *estimation* and *detection*.

- ▶ Estimation: finding a good estimate $\hat{g}(t)$ of an unknown signal $g(t)$ using information of other signals
- ▶ Detection: identifying which of one of a finite number of possible signals is present, again based on signals that depend on $g(t)$.

In communication systems, the available signal is the output of a channel:

$$z(t) = h(t) * v(t) + w(t),$$

where the channel attenuates and distorts the input and adds noise.

Component of a Vector along Another Vector

A simple example of a channel with noise results in output

$$\mathbf{g} = c\mathbf{x} + \mathbf{e},$$

where the gain c and error \mathbf{e} are unknown.

There are infinitely many solutions. To minimize error, we choose

$$\mathbf{e} = \mathbf{g} - c\mathbf{x}$$

to be orthogonal to \mathbf{x} .



Component of a Vector along Another Vector, II

The component of \mathbf{g} along \mathbf{x} is $\|\mathbf{g}\| \cos \theta$. Therefore

$$c\|\mathbf{x}\|^2 = \|\mathbf{x}\|\|\mathbf{g}\| \cos \theta = \langle \mathbf{g}, \mathbf{x} \rangle$$

So we can solve for c :

$$c = \frac{\langle \mathbf{g}, \mathbf{x} \rangle}{\|\mathbf{x}\|^2} = \frac{\langle \mathbf{g}, \mathbf{x} \rangle}{\langle \mathbf{x}, \mathbf{x} \rangle}$$

This answer applies to continuous-time signals. Suppose $g(t)$ is defined for $t_1 < t < t_2$. We can approximate $g(t)$ by $cx(t)$. The energy of the error

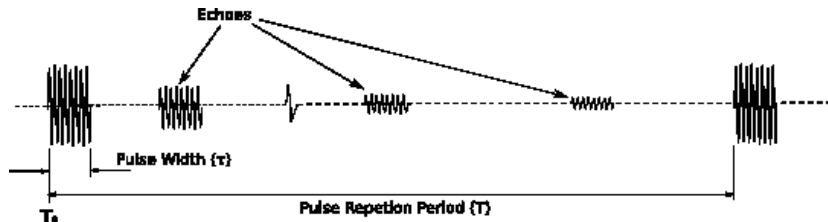
$$E_e = \int_{t_1}^{t_2} |g(t) - cx(t)|^2 dt$$

is minimized by solving $\partial E_e / \partial c = 0$:

$$c = \frac{\int_{t_1}^{t_2} g(t)x(t) dt}{\int_{t_1}^{t_2} |x(t)|^2 dt} = \frac{1}{E_x} \int_{t_1}^{t_2} g(t)x(t) dt$$

Application to Signal Detection

Consider a radar pulse $g(t)$.



The return signal depends on whether the transmit signal encounters a target. Here $w(t)$ represents noise and interference.

$$y(t) = \begin{cases} ag(t - t_0) & \text{target present} \\ w(t) & \text{target absent} \end{cases}$$

Typical pulse is $1 \mu\text{s}$ at 3 GHz; repetition rates from 2 kHz to 200 KHz

Application to Signal Detection, II

A key assumption (usually justified) is that noise $w(t)$ is *uncorrelated* with signal.

$$\langle w(t), g(t - t_0) \rangle = \int_{t_1}^{t_2} w(t)g(t - t_0) dt = 0$$

To see if a target is present at distance corresponding to delay t_0 , we *correlate* the return signal with a delayed version of the radar pulse:

$$\begin{aligned} \langle y(t), g(t - t_0) \rangle &= \begin{cases} \int_{t_1}^{t_2} a|g(t - t_0)|^2 dt & \text{target present} \\ 0 & \text{target absent} \end{cases} \\ &= \begin{cases} aE_g & \text{target present} \\ 0 & \text{target absent} \end{cases} \end{aligned}$$

Obviously, we decide based on whether the correlation is nonzero. To detect targets at all possible distances, we vary t_0 .