Signals and Systems: Part 2

- The Fourier transform in $2\pi f$
- Some important Fourier transforms
- Some important Fourier transform theorems
- Convolution and Modulation
- Ideal filters

Fourier transform definitions

- In EE 102A, the Fourier transform and its inverse were defined by

$$G(j\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt\,, \quad g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)e^{j\omega t}\,d\omega$$

  We used $j\omega$ to make all transforms similar in form (Fourier, Laplace, DTFT, z).

- In this class we use $2\pi f$ instead of ω. If we replace ω with $2\pi f$,

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\,dt\,, \quad g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\,df$$

  The transform and its inverse are almost symmetric; only the sign in the complex exponential changes.

- This will simplify a lot of the transforms and theorems.

Fourier transform existence

- If $g(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |g(t)| \, dt < \infty$$

  then $G(f)$ exists for every frequency f and is continuous.

- If $g(t)$ has finite energy, i.e.,

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty$$

  then $G(f)$ exists for "most" frequencies f and has finite energy.

- If $g(t)$ is periodic and has a Fourier series, then

$$G(f) = \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0)$$

  is a weighted sum of impulses in frequency domain.

Fourier transform examples

One-sided exponential decay is defined by $e^{-at}u(t)$ with $a > 0$:

$$g(t) = \begin{cases} 0 & t < 0 \\ e^{-at} & t > 0 \end{cases}$$

The Fourier transform of one-sided decay is (simple calculus):

$$\begin{aligned} G(f) &= \int_0^\infty e^{-at}\, e^{-j2\pi ft}\, dt \\ &= \int_0^\infty e^{-(a+j2\pi f)t}\, dt \\ &= \left[\frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)} \right]_{t=0}^{t=\infty} = \frac{1}{a+j2\pi f} \end{aligned}$$

Since $g(t)$ has finite area, its transform is continuous.

Even though $g(t)$ is real, its transform is complex valued. This is the usual situation.

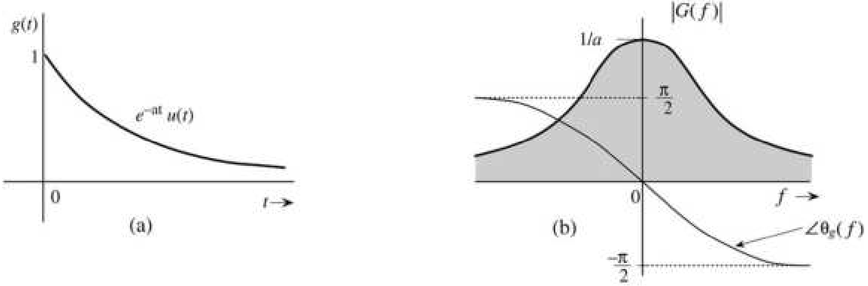
Fourier transform examples (cont.)

We can rationalize $G(f)$:

$$G(f) = \frac{1}{a + j2\pi f} = \frac{a - j2\pi f}{a^2 + 4\pi^2 f^2} = \frac{a}{a^2 + 4\pi^2 f^2} - j\frac{2\pi f}{a^2 + 4\pi^2 f^2}$$

We can better picture $G(f)$ using polar representation.

$$|G(f)| = \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}}, \quad \theta_G(f) = \angle G = -\tan^{-1}\left(\frac{-2\pi f}{a}\right)$$



Fourier transform examples (cont.)

Fourier transform at $f = 0$:

$$G(0) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi \cdot 0 \cdot t}\, dt = \int_{-\infty}^{\infty} g(t)\, dt$$

In general, $G(0)$ is the area under $g(t)$, called the *DC value*.

For $g(t) = e^{-at}u(t)$, $G(0) = 1/a$ is the only real value. It is also largest in magnitude.

For one-sided decay, $G(-f) = G^*(f)$, complex conjugate of $G(f)$.
This is true for all real-valued signals.

$$\begin{aligned} G^*(f) &= \left(\int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \right)^* = \int_{-\infty}^{\infty} g(t)^* (e^{-j2\pi f t})^*\, dt \\ &= \int_{-\infty}^{\infty} g(t) e^{j2\pi f t}\, dt = \int_{-\infty}^{\infty} g(t) e^{-j2\pi(-f)t}\, dt = G(-f) \end{aligned}$$

We need look at only positive frequencies for real-valued signals.

Duality

Fourier inversion theorem: if $g(t)$ has Fourier transform $G(f)$, then

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft}\, dt \;\text{ and }\; g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft}\, df$$

Inverse transform differs from forward transform only in sign of exponent.

Suppose we consider $G(\cdot)$ to be a function of t instead of f.
If we apply the Fourier transform again:

$$\int_{-\infty}^{\infty} G(t) e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} G(t) e^{j2\pi(-f)t}\, dt = g(-f)$$

We can summarize this as

$$\mathcal{F}\{g(t)\} = G(f) \;\Rightarrow\; \mathcal{F}\{G(t)\} = g(-f)$$

This is the *principle of duality*.

Important Fourier transforms

The unit rectangle function $\Pi(t)$ is defined by

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

Its Fourier transform is

$$\begin{aligned} \mathcal{F}\left\{\Pi(t)\right\} &= \int_{-\infty}^{\infty} e^{-i2\pi f t}\Pi(t)\,dt \\ &= \int_{-1/2}^{1/2} e^{-i2\pi f t}\,dt = \frac{\sin \pi f}{\pi f} = \operatorname{sinc}(\pi f) \end{aligned}$$

By duality, we know that

$$\mathcal{F}\{\operatorname{sinc}(\pi t)\} = \Pi(-f) = \Pi(f)$$

since $\Pi(f)$ is even. We say $\Pi(t)$ and $\operatorname{sinc}(\pi f)$ are a *Fourier pair*:

$$\Pi(t) \rightleftharpoons \operatorname{sinc}(\pi f)$$

Fact: every finite width pulse has a transform with unbounded frequencies.

Fourier transform time scaling

If $a > 0$ and $g(t)$ is a signal with Fourier transform $G(f)$, then

$$\mathcal{F}\{g(at)\} = \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty} g(u)e^{-j2\pi f(u/a)}\frac{du}{a} = \frac{1}{a}G\Big(\frac{f}{a}\Big)$$

If $a < 0$, change of variables requires reversing limits of integration:

$$\mathcal{F}\{g(at)\} = -\frac{1}{a}G\Big(\frac{f}{a}\Big)$$

Combining both cases:

$$\mathcal{F}\{g(at)\} = \frac{1}{|a|}G\Big(\frac{f}{a}\Big)$$

Special case: $a = -1$. The Fourier transform of $g(-t)$ is $G(-f)$.

Compressing in time corresponds to expansion in frequency (and reduction in amplitude) and vice versa.

The sharper the pulse the wider the spectrum.

Fourier transform time scaling example

The transform of a narrow rectangular pulse of area 1 is

$$\mathcal{F}\left\{\frac{1}{\tau}\Pi(t/\tau)\right\} = \operatorname{sinc}(\pi\tau f)$$

In the limit, the pulse is the unit impulse, and its tranform is the constant 1.

We can find the Fourier transform directly:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt = e^{-j2\pi f t}\Big|_{t=0} = 1$$

so

$$\delta(t) \rightleftharpoons 1$$

The impulse is the mathematical abstraction of signal whose Fourier transform has magnitude 1 and phase 0 for all frequencies.

By duality, $\mathcal{F}\{1\} = \delta(f)$. All DC, no oscillation.

$$1 \rightleftharpoons \delta(f)$$

Important Fourier transforms (cont.)

- Shifted impulse $\delta(t-t_0)$:

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j2\pi f t}\,dt = e^{-j2\pi f t_0}$$

  This is a complex exponential in frequency.

$$\delta(t-t_0) \rightleftharpoons e^{-j2\pi f t_0}$$

  This is an example of *shift theorem*: $\mathcal{F}\{f(t-t_0)\} = e^{-j2\pi f t_0}\mathcal{F}\{f(t)\}$.

  By duality, complex exponential in time has an impulse in frequency:

$$e^{j2\pi f_0 t} \rightleftharpoons \delta(f-f_0)$$

- Sinuoids: frequency content is concentrated at $\pm f_0$ Hz.

$$\begin{aligned}\mathcal{F}\{\cos 2\pi f_0 t\} &= \mathcal{F}\{\tfrac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\} \\ &= \tfrac{1}{2}\delta(f-f_0) + \tfrac{1}{2}\delta(f+f_0)\end{aligned}$$

$$\begin{aligned}\mathcal{F}\{\sin 2\pi f_0 t\} &= \mathcal{F}\{\tfrac{1}{2i}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\} \\ &= \tfrac{1}{2i}\delta(f-f_0) - \tfrac{1}{2i}\delta(f+f_0)\end{aligned}$$

Important Fourier transforms (cont.)

- Laplacian pulse $g(t) = e^{-a|t|}$ where $a > 0$. Since

  $$g(t) = e^{-at}u(t) + e^{at}u(-t)\,,$$

  we can use reversal and additivity:

  $$G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + 4\pi^2 f^2}\,.$$

  This is twice the real part of the Fourier transform of $e^{-at}u(t)$

- The signum function can be approximated as

  $$\operatorname{sgn}(t) = \lim_{a \to 0} \left(e^{-at}u(t) - e^{at}u(-t)\right)$$

  This has the Fourier transform

  $$\mathcal{F}\left\{\operatorname{sgn}(t)\right\} = \lim_{a \to 0} \left(\frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f}\right) = \frac{1}{j\pi f}$$

Important Fourier transforms (cont.)

- The unit step function is

$$u(t) = \tfrac{1}{2} + \tfrac{1}{2}\mathrm{sgn}(t)$$

  This has the Fourier transform

$$\mathcal{F}\{u(t)\} = \tfrac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

Fourier transform properties

- Time delay causes linear phase shift.

$$\begin{aligned}\mathcal{F}\left\{g(t-t_0)\right\} &= \int_{-\infty}^{\infty} g(t-t_0)e^{-j2\pi ft}\,dt \\ &= \int_{-\infty}^{\infty} g(u)e^{-j2\pi f(u+t_0)}\,du \\ &= e^{-j2\pi ft_0}\int_{-\infty}^{\infty} g(u)e^{-j2\pi fu}\,du = e^{-j2\pi ft_0}G(f)\end{aligned}$$

  Therefore

$$g(t-t_0) \rightleftharpoons e^{-j2\pi ft_0}G(f)$$

- By duality we get frequency shifting (modulation):

$$e^{j2\pi f_c t}g(t) \rightleftharpoons G(f-f_c)$$

$$\cos(2\pi f_c t)g(t) \rightleftharpoons \tfrac{1}{2}G(f+f_c) + \tfrac{1}{2}G(f-f_c)$$

Fourier transform properties (cont.)

- Convolution in time. The convolution of two signals is

$$g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(u) g_2(t-u)\, du$$

- The Fourier transform of the convolution is the product of the transforms.

$$g_1(t) * g_2(t) \rightleftharpoons G_1(f) G_2(f)$$

  The Fourier transform reduces convolution to a simpler operation.

- Note that there is no factor of $\frac{1}{2\pi}$ for frequency domain convolution, as there was in EE 102A.

Fourier transform properties (cont.)

- Multiplication in time.

$$g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)\,d\lambda$$

- Again, there is no $\frac{1}{2\pi}$ factor, as there was in EE 102A.
- Convolution in one domain goes exactly to multiplication in the other domain, and multiplication to convolution.
- The modulation theorem is a special case.

$$\begin{aligned}\mathcal{F}\left\{g(t)\cos(2\pi f_c t)\right\} &= \mathcal{F}\left\{g(t))\right\} * \mathcal{F}\left\{\cos(2\pi f_c t)\right\} \\ &= G(f) * \left(\tfrac{1}{2}\delta(f+f_0) + \tfrac{1}{2}\delta(f-f_0)\right) \\ &= \tfrac{1}{2}G(f+f_c) + \tfrac{1}{2}G(f-f_c)\end{aligned}$$

The triangle function $\Delta(t)$ and its Fourier transform

- The book defines the triangle function $\Delta(t)$ as

$$\Delta(t) = \begin{cases} 1 - 2|x| & |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

  This is another unfortunate choice, but not as bad as $\operatorname{sinc}(t)$!

- The triangle function can be written as twice the convolution of two rectangle functions of width $\frac{1}{2}$.

$$\Delta(t) = 2\ \Pi(2t) * \Pi(2t)$$

  where the factor of 2 is needed to make the convolution 1 at $t = 0$.

- The Fourier transform is therefore

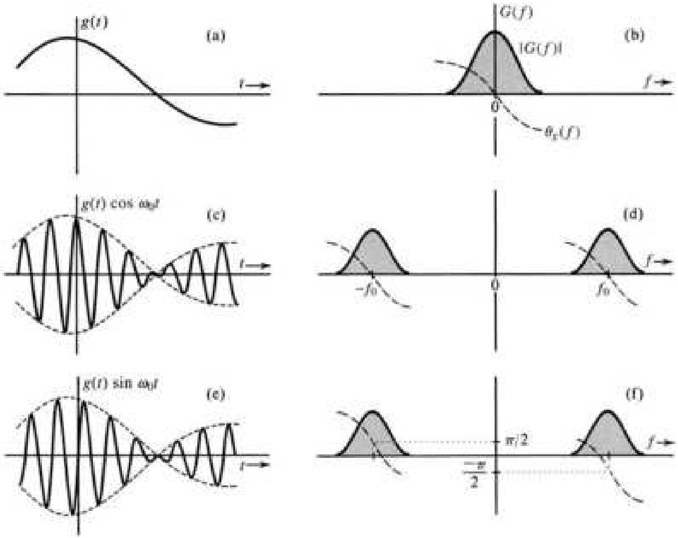
$$\begin{aligned} \mathcal{F}\left\{\Delta(t)\right\} &= \mathcal{F}\left\{2\Pi(2t) * \Pi(2t)\right\} = 2\mathcal{F}\Big(\left\{\Pi(2t)\right\}\Big)^2 \\ &= 2\left(\tfrac{1}{2}\operatorname{sinc}\left(\tfrac{\pi}{2}f\right)\right)^2 = \tfrac{1}{2}\operatorname{sinc}^2\left(\tfrac{1}{2}\pi f\right) \end{aligned}$$

  Then

$$\Delta(t) \rightleftharpoons \tfrac{1}{2}\operatorname{sinc}^2(\tfrac{1}{2}\pi f)$$

Modulation theorem

Modulation of a baseband signal creates replicas at $\pm$ the modulation frequency.



Applications of modulation

- For transmission by radio, antenna size is proportional to wavelength.

  Low frequency signals (voice, music) must be converted to higher frequency.

- To share bandwidth, signals are modulated by different carrier frequencies.
  - North America AM radio band: 535–1605 KHz (10 KHz bands)
  - North America FM radio band: 88–108 MHz (200 KHz bands)
  - North America TV bands: VHF 54–72, 76–88, 174–216, UHF 470–806, 806–890

  Frequencies can be reused in different geographical areas.

  With digital TV, channel numbers do not correspond to frequencies.

Rats laughing: `http://www.youtube.com/watch?v=j-admRGFVNM`

Bandpass signals

Bandlimited signal: $G(f) = 0$ if $|f| > B$.

Every sinusoid $\sin(2\pi f_c t)$ has bandwidth f_c.

If $g_c(t)$ and $g_s(t)$ are bandlimited, then

$$m(t) = g_c(t)\cos(2\pi f_c t) + g_s(t)\sin(2\pi f_c t)$$

is a *bandpass* signal. Its Fourier transform or spectrum is restricted to

$$f_c - B < |f| < f_c + B$$

The bandwidth is $(f_c + B) - (f_c - B) = 2B$.

Most signals of interest in communications will be either bandpass (RF), or baseband (ethernet).

Filters

A *filter* is a system that modifies an input. (E.g., an optical filter blocks certain frequencies of light.)

In communication theory, we usually consider linear filters, which are linear time-invariant systems.

$$L(v_1(t) + v_2(t)) = L(v_1(t)) + L(v_2(t))$$
$$L(av(t)) = aL(v(t))$$
$$L(v(t - t_0)) = (Lv)(t - t_0)$$

Fundamental fact: every LTIS is defined by convolution:

$$Lv(t) = h(t) * v(t) = \int_{-\infty}^{\infty} h(u)v(t-u)\,du = \int_{-\infty}^{\infty} h(t-u)v(u)\,du$$

The signal $h(t)$ is called the *impulse response* because

$$L\delta(t) = \int_{-\infty}^{\infty} \delta(u)h(t-u)\,du = h(t)$$

Transfer function

By the convolution theorem,

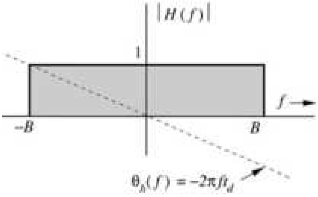
$$h(t) * v(t) \rightleftharpoons H(f)V(f)$$

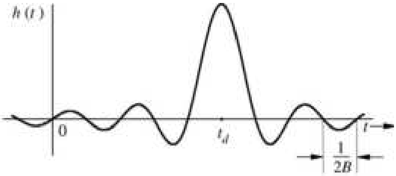
$H(f)$ is called the *transfer function*.

Many systems are best understood in the frequency domain.

Example: low pass filter with cutoff frequency B:

$$H(f) = \Pi\left(\frac{f}{2B}\right)e^{-j2\pi f t_d} \rightleftharpoons 2B\,\mathrm{sinc}(2\pi B(t - t_d))$$

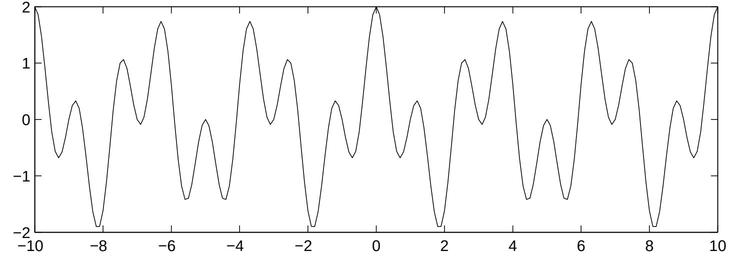


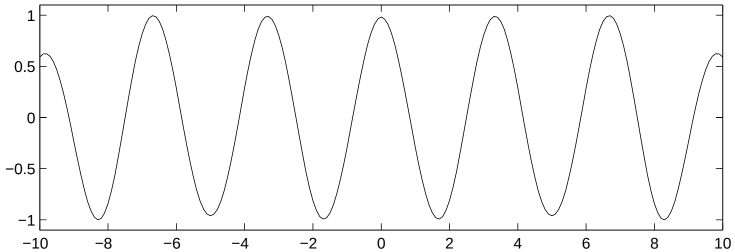


Note that the system is not (cannot) be causal.

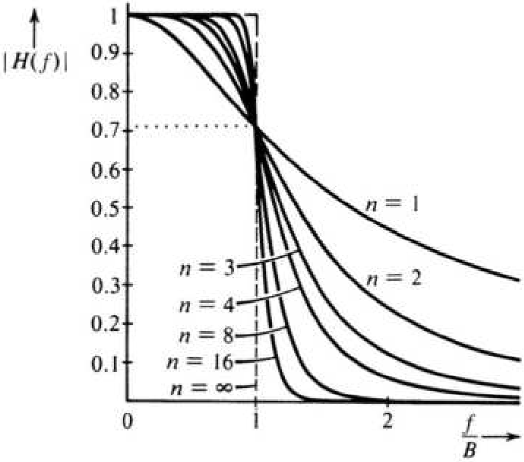
Low-pass filter example

$x(t) = \cos(2\pi(0.3)t) + \cos(2\pi(0.8)t),\ h(t) = \mathrm{sinc}(\pi t),\ y(t) = \cos(2\pi(0.3)t)$



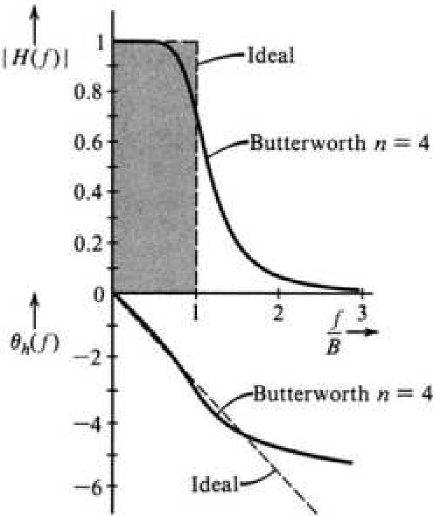


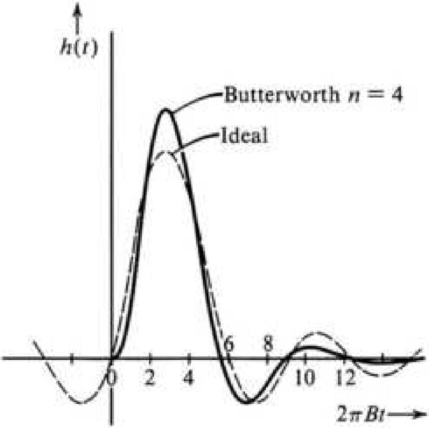
Butterworth filter: nonideal low-pass filter



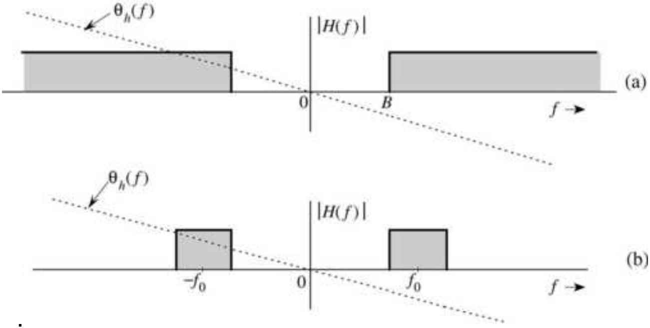
Butterworth filter has $|H(f)| = \dfrac{1}{\sqrt{(f/B)^{2n}}}$

Butterworth filter vs. ideal low-pass filter





High-pass and band-pass filters



If the phase is zero

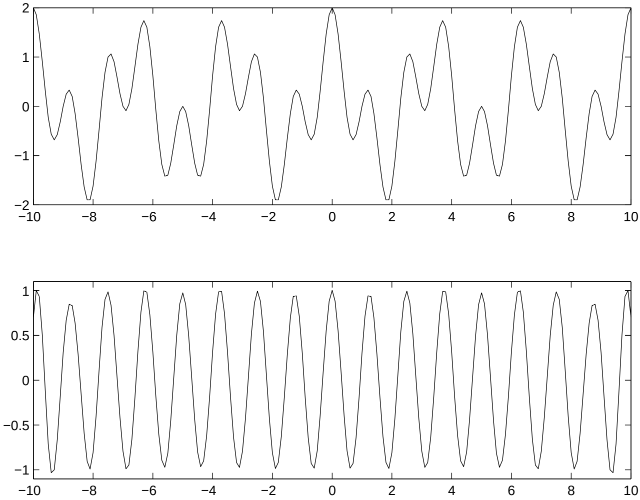
$$h_{\text{highpass}}(t) = \delta(t) - 2B\,\mathrm{sinc}(2\pi Bt)$$

$$h_{\text{bandpass}}(t) = 2\cos(2\pi f_0 t)\,2B\,\mathrm{sinc}(2\pi Bt)$$

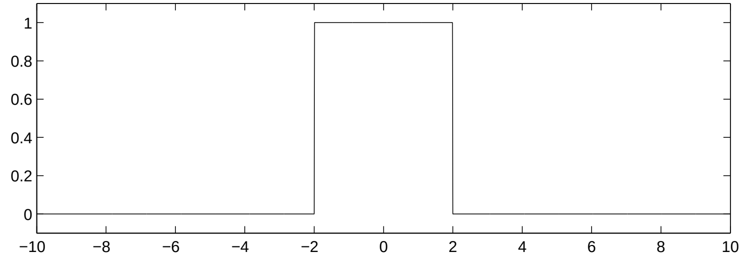
In practice there will be an additional linear phase in the spectral profile, corresponding to a delay in time. This makes the filters realizable.

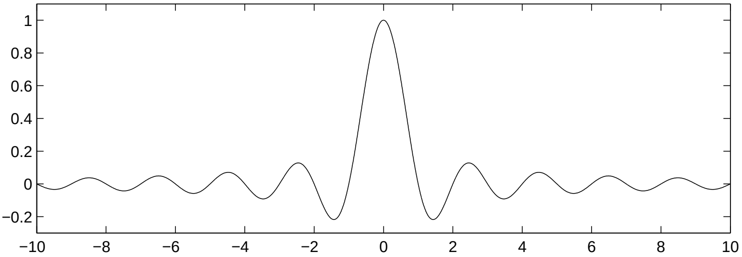
Band-pass filter example

$x(t) = \cos(2\pi(0.3)t) + \cos(2\pi(0.8)t)$, $h(t) = \cos 2\pi t \; \mathrm{sinc}\, \pi t$,
$y(t) = \cos(2\pi(0.8)t)$

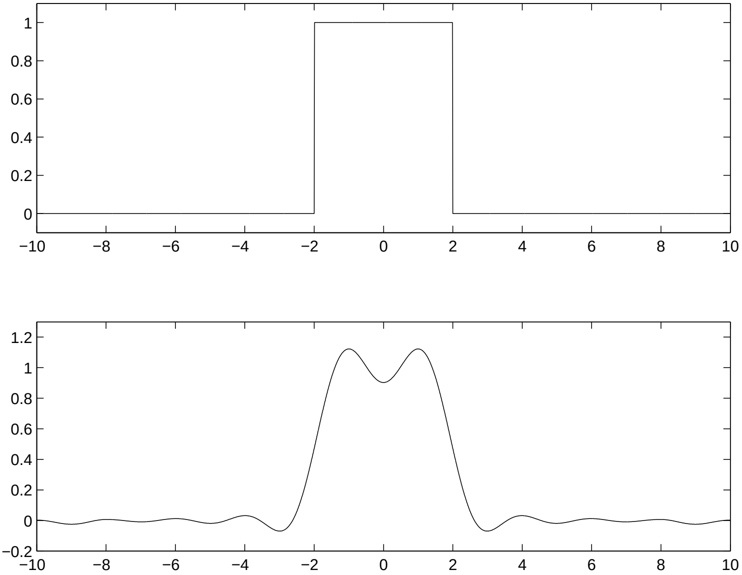


Low-pass filter: $x(t) = \Pi(\frac{t}{4})$, $h(t) = \operatorname{sinc} t$

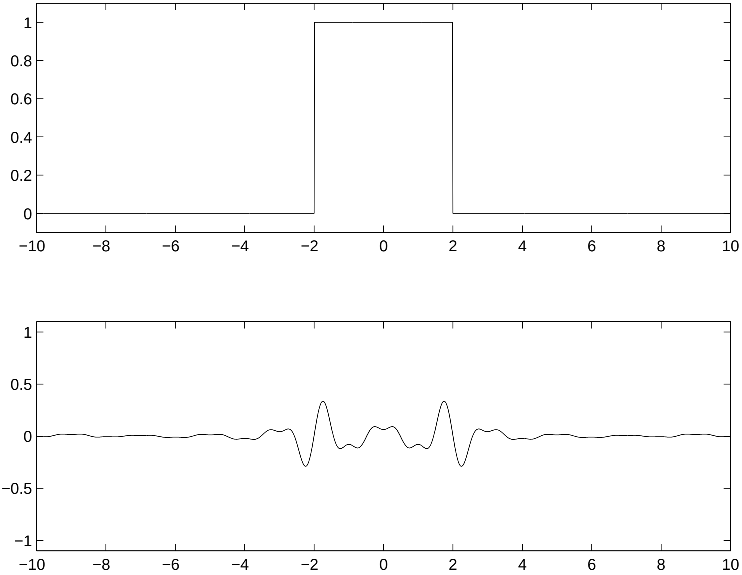




Low-pass filter: $x(t) = \Pi(\frac{t}{4})$, $h(t) = \operatorname{sinc} t$



Band-pass filter: $x(t) = \Pi(\frac{t}{4})$, $h(t) = \cos 2\pi t \, \operatorname{sinc} t$



Examples of Communication Channels

- wires (PCD trace or conductor on IC)
- optical fiber (attenuation 4dB/km)
- broadcast TV (50 kW transmit)
- voice telephone line (under -9 dbm or 110 μW)
- walkie-talkie: 500 mW, 467 MHz
- Bluetooth: 20 dBm, 4 dBm, 0 dBm
- Voyager: X band transmitter, 160 bit/s, 23 W, 34m dish antenna

`http://science.time.com/2013/03/20/humanity-leaves-the-solar-system-35-years-later-voyager-offically-exits-the-heliosphere/`

Communication Channel Distortion

The linear description of a channel is its impulse response $h(t)$ or equivalently its transfer function $H(f)$.

$$y(t) = h(t) * x(t) \quad \rightleftharpoons \quad Y(f) = H(f)X(f)$$

Note that $H(f)$ both attenuates ($|H(f)|$) and phase shifts ($\angle H(f)$) the input signal.

Channels are subject to impairments:

- Nonlinear distortion (e.g., clipping)
- Random noise (independent or signal dependent)
- Interference from other transmitters
- Self interference (reflections or multipath)

Channel Equalization

Linear distortion can be compensated for by *equalization*.

$$H_{\mathrm{eq}}(f) = \frac{1}{H(f)} \Rightarrow \hat{X}(f) = H_{\mathrm{eq}}(f)Y(f) = X(f)$$

The equalization filter accentuates frequencies that are attenuated by the channel.

However, if $y(t)$ includes noise or interference,

$$y(t) = x(t) + z(t)$$

then

$$H_{\mathrm{eq}}(f)Y(f) = X(f) + \frac{Z(f)}{H(f)}$$

Equalization may accentuate noise!

Next time

- Lab this Friday : How do you figure out where signals are? Spectrograms and waterfall plots
- Next class Monday : 3.6 – 3.8 in Lathi and Ding. Signal distortion, power spectral density, correlation and autocorrelation.
- Wednesday : Begin Chapter 4. Analog modulation schemes.
- Lab next Friday : finding and decoding airband AM