Angle Modulation, II

Lecture topics

- FM bandwidth and Carson's rule
- Spectral analysis of FM
- Narrowband FM Modulation
- Wideband FM Modulation

Bandwidth of Angle-Modulated Waves

Angle modulation is nonlinear and complex to analyze.

Early developers thought that bandwidth could be reduced to 0.
They were wrong. FM has infinite bandwidth.

Two approximations for FM:

- narrowband approximation (NBFM)
- wideband approximation (WBFM)

NBFM: if $a(t) = \int m(u)\,du$ and $|k_f a(t)| \ll 1$, then $B_s \approx 2B_m$ and

$$\varphi_{\mathrm{FM}}(t) \approx A\big(\cos\omega_c t - k_f a(t)\sin\omega_c t\big)$$

WBFM: if $\Delta f = \max|k_f m(t)|$ is peak frequency deviation, then

$$B_s = 2\Delta f + 2B_m$$

This is known as known as *Carson's rule*.

J.R. Carson, Proc. IRE, 1922.

Narrowband FM

Let $a(t) = \int_{-\infty}^{t} m(u)\,du$. Let's define the *complex FM signal* by:

$$\hat{\varphi}_{\mathrm{FM}}(t) = Ae^{j(\omega_c t + k_f a(t))} = Ae^{jk_f a(t)} \cdot e^{j\omega_c t}$$

Then, $\varphi_{\mathrm{FM}}(t) = \mathrm{Re}\left(\hat{\varphi}_{\mathrm{FM}}(t)\right)$.

Maclaurin power series expansion of $\hat{\varphi}_{\mathrm{FM}}(t)$:

$$A\left(1 + jk_f a(t) - \frac{k_f^2}{2!}a^2(t) + \cdots + j^n \frac{k_f^n}{n!}a^n(t) + \cdots\right)e^{j\omega_c t}$$

If $a(t)$ has a bandwidth $2B$ Hz, then the n-th term has a bandwidth $n2B$.

This expansion for $\hat{\varphi}_{\mathrm{FM}}(t)$ shows that the bandwidth is infinite.

However, things aren't quite that bad . . .

Since $k_f^n/n! \to 0$, all but a small amount of power is in a finite band.

Maclaurin series is Taylor series centered at zero.

Narrowband FM (cont.)

Using $\varphi_{\mathrm{FM}}(t) = \mathrm{Re}\left(\hat{\varphi}_{\mathrm{FM}}(t)\right)$, the FM signal is

$$A\Big(\cos\omega_c t - k_f a(t)\sin\omega_c t - \frac{k_f^2}{2!}a^2(t)\cos\omega_c t + \frac{k_f^3}{3!}a^3(t)\sin\omega_c t + \cdots\Big)$$

If $|k_f a(t)| \ll 1$ then all but first two terms are negligible.

The narrowband FM approximation is

$$\varphi_{\mathrm{FM}}(t) \approx A\big(\cos\omega_c t - k_f a(t)\sin\omega_c t\big)$$

NBFM signal has bandwidth $2B$, same as bandwidth of AM.

NBFM has power $\frac{1}{2}A^2$, which does not depend directly on $m(t)$.

Similarly, the narrowband approximation for phase modulation is

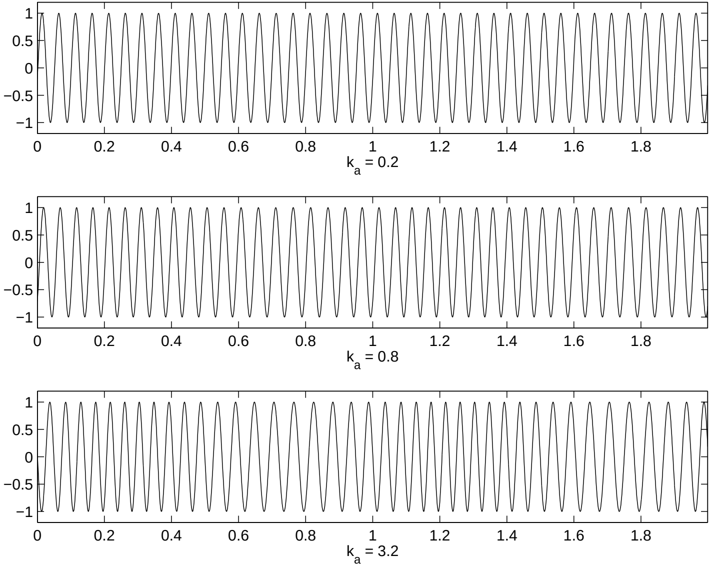
$$\varphi_{\mathrm{PM}}(t) \approx A\big(\cos\omega_c t - k_p m(t)\sin\omega_c t\big)$$

SNR of NBFM will be discussed later.

The narrowband approximation is a special case of *linearization* — finding the linear approximation to a function at a given point.

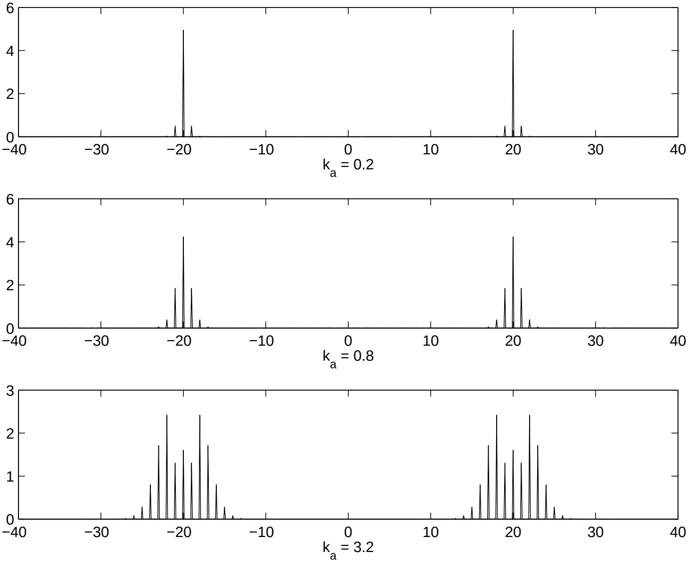
Frequency Modulation of Tone: $f_c = 20,\ f_m = 1$

$$\varphi_{\mathrm{FM}}(t) = \cos\left(2\pi f_c + k_a a(t)\right),\ |k_f a(t)| = 0.2, 0.8, 3.2$$



Fourier Transforms of Tone FM

$$\varphi_{\mathrm{FM}}(t) = \cos\left(2\pi f_c + k_a a(t)\right),\ |k_f a(t)| = 0.2, 0.8, 3.2$$



Wideband FM (WBFM) Bandwidth

For wideband FM, the frequency deviation contributes to FM bandwidth.

The FM signal for message signal $m(t)$ is

$$\varphi_{\mathrm{FM}}(t) = \cos(\omega_c + k_f a(t)) , \quad \text{where} \quad a(t) = \int_{\infty}^{t} m(\tau) d\tau .$$

If $m(t)$ has bandwidth $2B$ Hz, then $a(t)$ also has bandwidth $2B$ Hz.

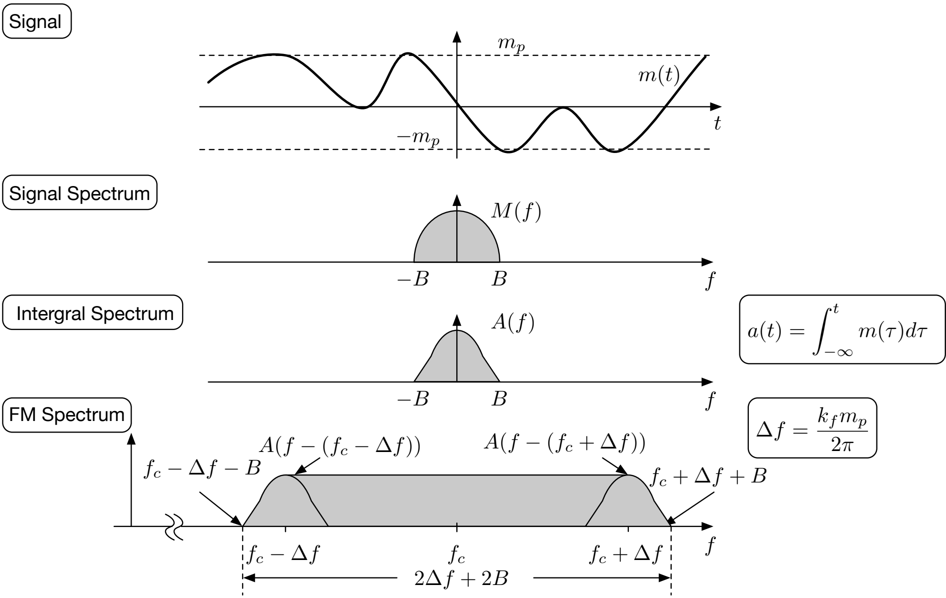
What is the bandwidth of $\varphi_{\mathrm{FM}}(t)$? This is a difficult question in general. There are explicit solutions for only a few signals, such as sinusoids.

In practice, there are two contributors to the bandwidth:

- Signal bandwidth $2B$
- FM deviation frequency $\Delta f = \dfrac{k_f m_p}{2\pi}$

This leads to Carson's rule: $B_{\mathrm{FM}} = 2\Delta f + 2B$

Carson's Rule in One Figure



Frequency Modulation of Tone

Spectral analysis of FM is difficult/impossible for general signals.

Consider sinusoidal input $m(t) = \cos \omega_m t \implies B_m = f_m = \frac{1}{2\pi}\omega_m$.

$$a(t) = \int_0^t m(u)\,du = \frac{1}{\omega_m} \sin \omega_m t$$

$$\hat{\varphi}_{\mathrm{FM}} = A \exp\left(j\omega_c t + \frac{k_f}{\omega_m} \sin \omega_m t \right) = A e^{j\omega_c t}\, e^{j\beta \sin \omega_m t}$$

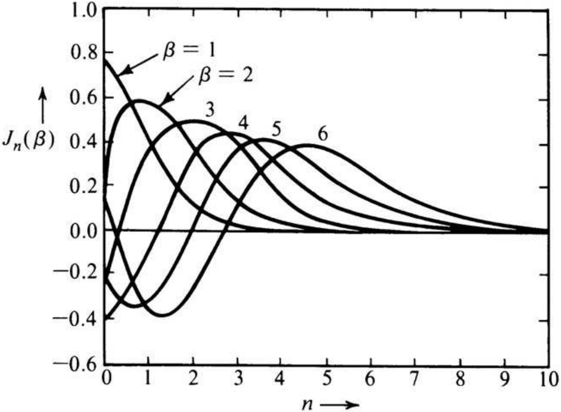
where $\beta = k_f \,/\, \omega_m$ is *frequency deviation ratio* or *FM modulation index*.

Since $e^{j\beta \sin \omega_m t}$ is periodic with frequency ω_m,

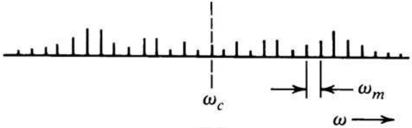
$$J_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin t - nt)}\, dt \implies e^{j\beta \sin \omega_m t} = \sum_n J_n(\beta) e^{jn\omega_m t}$$

J_n is a Bessel function. $J_n(\beta)$ is negligible if $n > \beta + 1$.

Bessel Function $J_n(\beta)$



(a)



(b)

US Broadcast FM

- Frequency range: $88.0 - 108.0$ MHz
- Channel width: 200 KHz (100 channels)
- Channel center frequencies: $88.1, 88.3, \ldots, 107.9$
- Frequency deviation: ± 75 KHz
- Signal bandwidth: high-fidelity audio requires only ± 20 KHz, so bandwidth is available for other applications:
  - Muzak (elevator music) (1936)
  - Stock market quotations
  - Interactive games
- Stereo uses sum and difference of L/R audio channels

FM radio was assigned the 42–50 MHz band of the spectrum in 1940. In 1945, at the behest of RCA (David Sarnoff CEO), the FCC moved FM to 88–108 MHz, obsoleting all existing receivers.

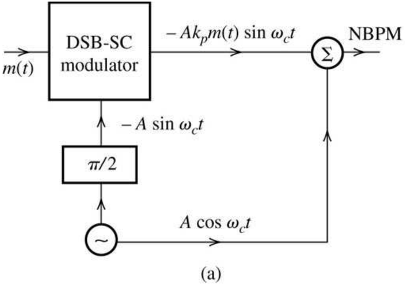
NBFM Modulation

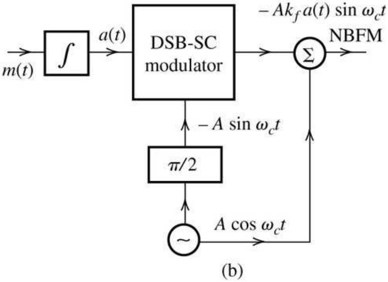
For narrowband signals, $|k_f a(t)| \ll 1$ and $|k_p m(t)| \ll 1$,

$$\hat{\varphi}_{\text{NBFM}} \approx A(\cos\omega_c - k_f a(t)\sin\omega_c t)$$

$$\hat{\varphi}_{\text{NBPM}} \approx A(\cos\omega_c - k_p m(t)\sin\omega_c t)$$

We can modulate using a DSB-SC modulator with a phase shifter.





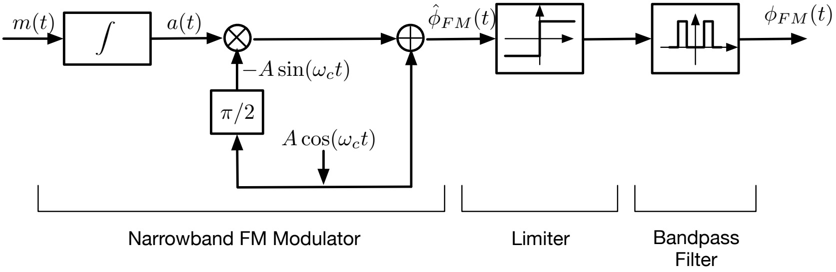
Phase modulation

Frequency modulation

NBFM Block Diagram

In practice, this modulation will not be perfect, and there will be some amplitude modulation remaining.

To fix this, follow with a limiter and a bandpass filter. For FM,

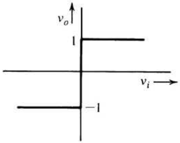


NBFM: Bandpass Limiter

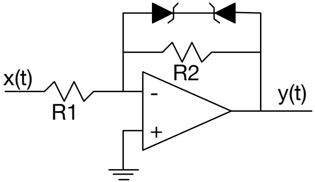
The input-output diagram for an ideal hard limiter is

$$v_o(t) = \begin{cases} +1 & v_i(t) > 0 \\ -1 & v_i(t) < 0 \end{cases}$$

This is a signum function — the output of a comparison against 0.



A hard limiter can be implemented by an op amp inverting amplifier, with back-to-back zener diodes to limit the output amplitude.



NBFM: Bandpass Limiter (cont.)

Input to bandpass limiter is

$$v_i(t) = A(t)\cos\theta(t)\,,\ \text{where } \theta(t) = \omega_c t + k_f \int_{-\infty}^{t} m(u)\,du$$

Ideally, $A(t)$ is constant, but it may vary slowly. We assume $A(t) > 0$. The input to the bandpass filter is

$$v_o(\theta) = \begin{cases} +1 & \cos\theta > 1 \\ -1 & \cos\theta < 1 \end{cases}$$

which is periodic in θ with period 2π. Its Fourier series is

$$\begin{aligned} v_o(\theta) &= \frac{4}{\pi}\left(\cos\theta - \tfrac{1}{3}\cos 3\theta + \tfrac{1}{5}\cos 5\theta + \cdots\right) \\ &= \frac{4}{\pi}\Big(\cos\Big(\omega_c t + k_f \int m(u)\,du\Big) - \tfrac{1}{3}\cos 3\Big(\omega_c t + k_f \int m(u)\,du\Big) + \cdots\Big) \end{aligned}$$

The bandpass filter eliminates all but the first term.

WBFM Modulation: Direct Generation Using VCO

A voltage controlled oscillator (VCO) generates a signal whose instantaneous frequency is proportional to an input $m(t)$:

$$\omega_i(t) = \omega_c + k_f m(t)$$

This signal is bandpass filtered, then used in a modulator.

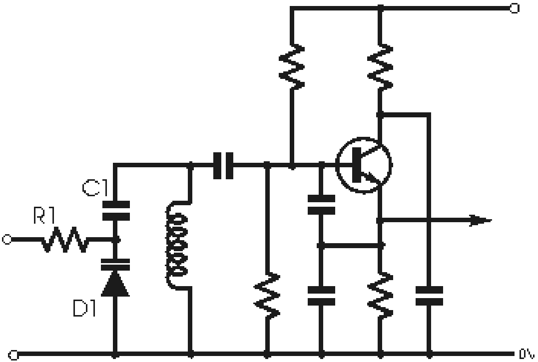
VCO can be constructed by using input voltage to control one or more circuit parameters:

- R: transistor with controlled gate voltage
- L: saturable core reactor
- C: reverse-biased semiconductor diode

In all cases, feedback is used to adjust the frequency.

VCO Using Varactor

The inductor is part of an LRC tank circuit whose frequency is determined by the capacitance on D1, which depends on the input voltage.



This is an example of a Hartley oscillator. There are lots of other circuits that have been invented for FM modulation.

FM Direct Digital Synthesis

Currently, it is very common to synthesize the FM waveform digitally either at an intermediate frequency f_i and then mix it up to the desired carrier, or to directly synthesize the waveform at the carrier frequency.

This is Direct Digital Synthesis, or DDS.

Typical intermediate frequencies are a few MHz, so the calculations are accurate but manageable, and undesired sidebands can be easily suppressed.