Sampling and Pulse Trains

- Sampling and interpolation
- Practical interpolation
- Pulse trains
- Analog multiplexing

Sampling Theorem

Sampling theorem: a signal $g(t)$ with bandwidth B can be reconstructed *exactly* from samples taken at any rate $R > 2B$.

Sampling can be achieved mathematically by multiplying by an impulse train. The unit *impulse train* is defined by

$$\text{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t-k)$$

The unit impulse train is also called the III or comb function.

Sampling a signal $g(t)$ uniformly at intervals T_s yields

$$\overline{g}(t) = g(t)\,\text{III}_{T_s}(t) = \sum_{n=-\infty}^{\infty} g(t)\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s)$$

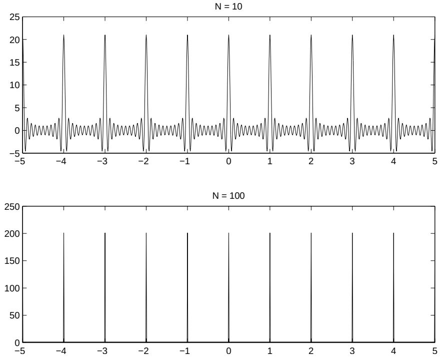
Only information about $g(t)$ at the sample points is retained.

Fourier Transform of $\text{III}(t)$

Fact: the Fourier transform of $\text{III}(t)$ is $\text{III}(f)$.

$$\mathcal{F}\,\text{III}(t) = \sum_{n=-\infty}^{\infty} \mathcal{F}\delta(t-n) = \sum_{n=-\infty}^{\infty} e^{-j2\pi nf} = \sum_{n=-\infty}^{\infty} e^{j2\pi nf} = \text{III}(f)$$

The complex exponentials cancel at noninteger frequencies and add up to an impulse at integer frequencies.



Fourier Transform of Sampled Signal

The impulse train $\text{III}(t/T_s)$ is periodic with period T_s.

$\text{III}(t/T_s)$ can be represented as sum of complex exponentials of multiples of the fundamental frequency:

$$\text{III}(t/T_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t} \quad (f_s = \frac{1}{T_s})$$

Thus

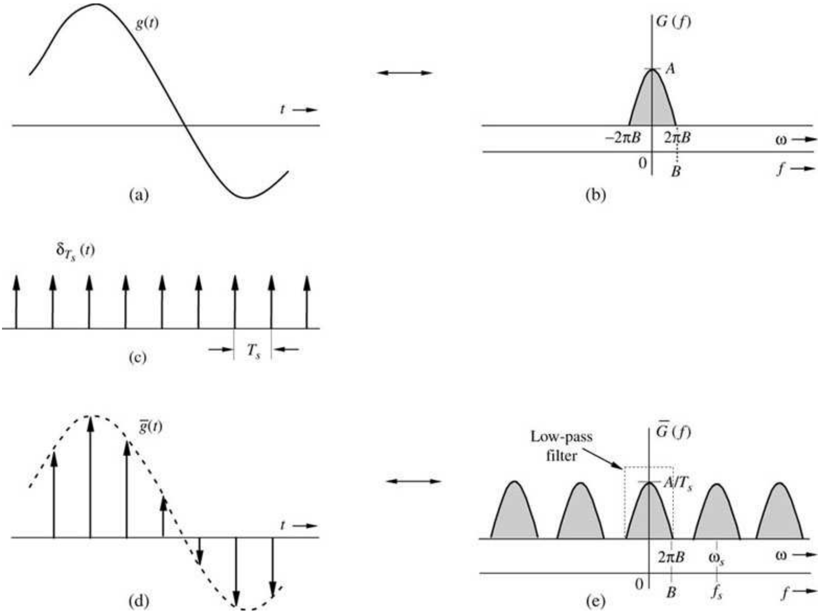
$$\overline{g} = g(t)\,\text{III}(t/T_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} g(nT_s)e^{j2\pi n f_s t}$$

and by the frequency shifting property

$$\overline{G}(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} G(f - nf_s)$$

This sum of shifts of the spectrum can be written as $\text{III}(f/f_s) * G(f)$.

Sampled Signal and Fourier Transform



Reconstruction from Uniform Samples (Ideal)

If sample rate $1/T_s$ is greater than $2B$, shifted copies of spectrum do not overlap, so low pass filtering recovers original signal.

Cutoff frequency of low pass filter should satisfy

$$B \leq f_c \leq f_s - B$$

Suppose $f_c = B$. A low pass filter with gain T_s has transfer function and impulse response

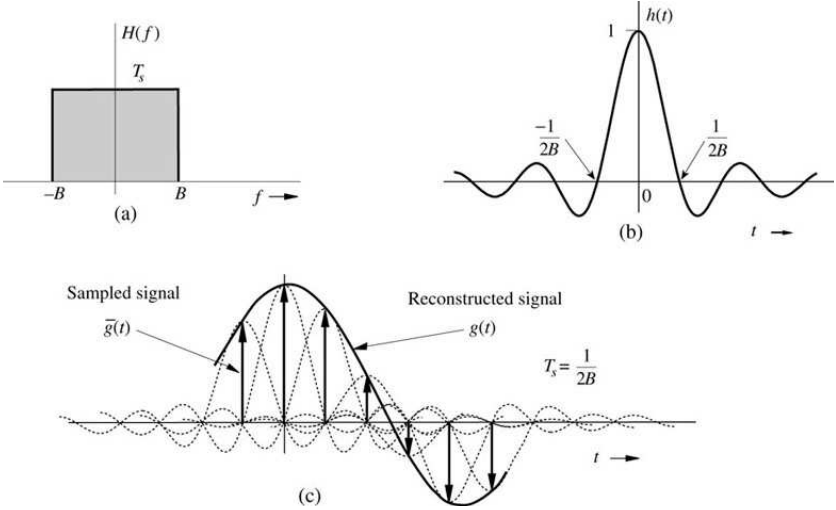
$$H(f) = T_s \Pi\left(\frac{f}{2B}\right), \quad h(t) = 2BT_s \operatorname{sinc}(2\pi Bt)$$

Then if $T_s = 1/2B$

$$\begin{aligned} h(t) * \overline{g}(t) &= \sum_{n=-\infty}^{\infty} h(t) * g(nT_s)\delta(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} g(nT_s) \operatorname{sinc}(2\pi B(t - nT_s)) \end{aligned}$$

Ideal Interpolation

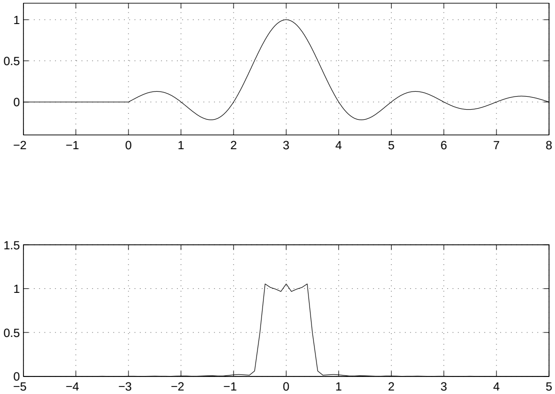
Ideal interpolation represents a signal as sum of shifted sincs.



Practical Interpolation

In practice we require a causal filter. We can delay the impulse response and eliminate values at negative times.

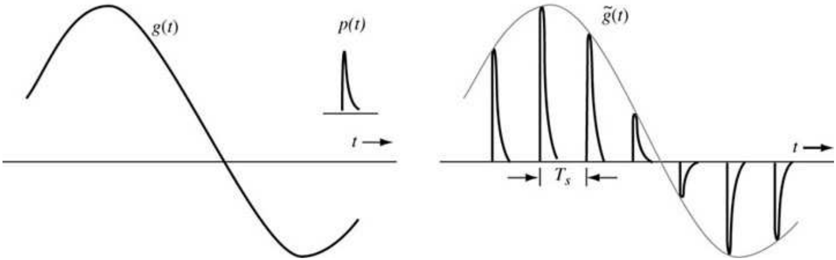
$$\tilde{h}(t) = \begin{cases} h(t - t_0) & t > 0 \\ 0 & t < 0 \end{cases}$$



Practical Interpolation (cont.)

In practice, the sampled signal is a sum of pulses, not impulses.

$$
\begin{aligned}
\tilde{g}(t) &= \sum_{n=-\infty}^{\infty} g(nT_s)p(t-nT_s) \\
&= p(t) * \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s) = p(t) * \overline{g}(t)
\end{aligned}
$$



Practical Interpolation (cont.)

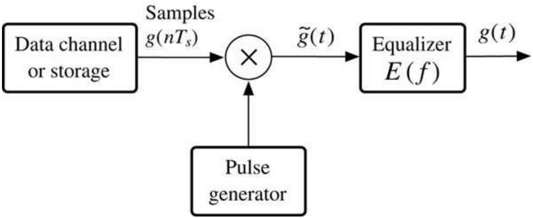
By the convolution theorem,

$$\tilde{G}(f) = P(f) \cdot \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(f - nf_s)$$

We can recover $G(f)$ from $\tilde{G}(f)$ by low pass filtering to eliminate high frequency shifts and *equalizing* by inverting $P(f)$.

$$E(f) = \begin{cases} T_s/P(f) & |f| < B \\ 0 & |f| > B \end{cases}$$

The transfer function $E(f)$ should not be close to 0 in the pass band.



Practical Interpolation (cont.)

Example: rectangular pulses with $T_p < T_s < 1/2B$.

$$p(t) = \Pi\left(\frac{t - 0.5T_p}{T_p}\right) \implies P(f) = T_p \operatorname{sinc}(\pi T_p f) e^{-j\pi T_p f}$$

The transfer function for the equalizer should satisfy

$$E(f) = \begin{cases} T_s/P(f) & |f| < B \\ \text{whatever} & B < |f| < 1/T_s - B \\ 0 & |f| > 1/T_s - B \end{cases}$$

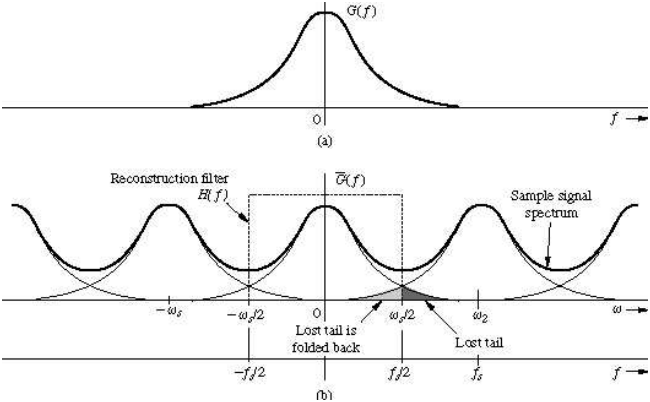
To avoid large gain (and noise amplification), we need $|P(f)|$ bounded away from 0. If $|T_p f| < 1$ then

$$\operatorname{sinc}(\pi T_p f) > 0$$

If $T_p < 1/2B$ then $P(f) > \sin(\pi/2)/(\pi/2) = 2/\pi$.

The Treachery of Aliasing

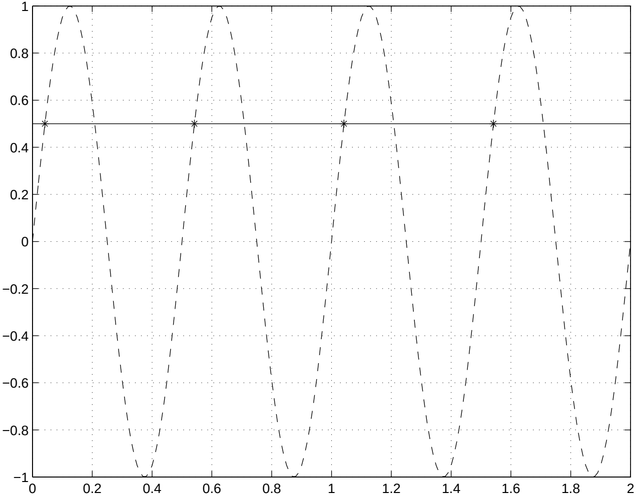
If we sample too slowly, the shifted spectrums overlap.



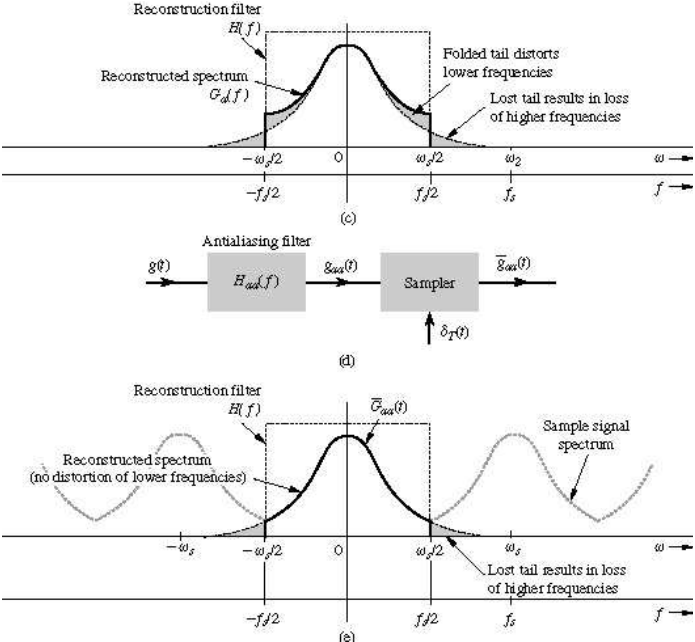
High frequency components are “folded” back into the spectrum. This should be avoided.

Example of Aliasing

$\cos 4\pi t$ sampled at 2 Hz looks like a constant.



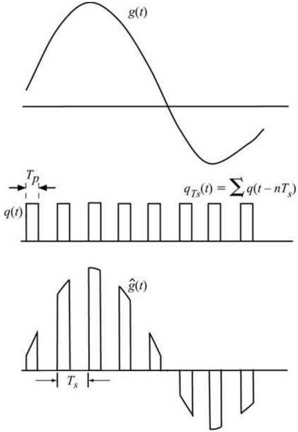
Anti-Aliasing Filter



Nonideal Practical Sampling

A real-world sampler cannot obtain the value of the signal at an instant. The sampling circuit measures the signal by integration.

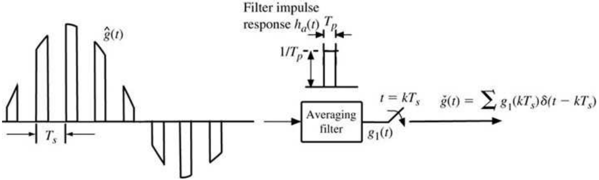
$$g_1(nT_s) = \int_{-T_p/2}^{T_p} q(t)g(t - nT_s)\, dt$$



Nonideal Practical Sampling (cont.)

The values obtained by averaging create a shaped impulse train:

$$\tilde{g}(t) = \sum_{n=-\infty}^{\infty} g_1(nT_s)\delta(t - nT_s)$$



Gating with a rectangle corresponds to transfer function

$$H_a(F) = \mathrm{sinc}(\pi T_p f)$$

Thus

$$G_1(f) = H(f) \sum_{n=-\infty}^{\infty} Q_n G(f - nf_s) = \mathrm{sinc}(\pi T_p f) \sum_{n=-\infty}^{\infty} Q_n G(f - nf_s)$$

Nonideal Practical Sampling (cont.)

We can use the sampling theorem to obtain

$$\tilde{G}(f) = \sum_{n=-\infty}^{\infty} F_n(f) G_1(f + n f_n)$$

where

$$F_n(f) = \frac{1}{T_s} \sum_{\ell=-\infty}^{\infty} Q_n \operatorname{sinc}((\pi f + (\ell + n)\pi f_s) T_p)$$

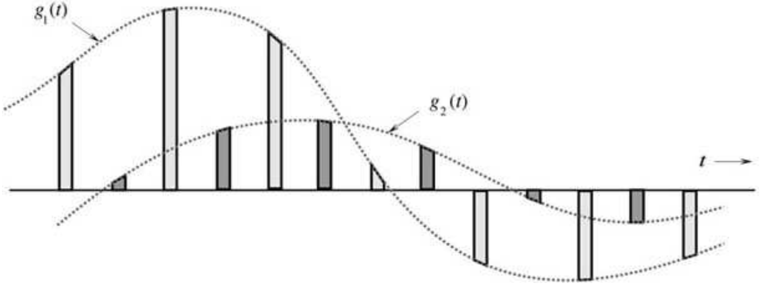
Low pass filtering $\tilde{g}(t)$ yields distorted signal with transform $F_0(f)G(f)$.

Original signal can be recoverd by equalizer filter.

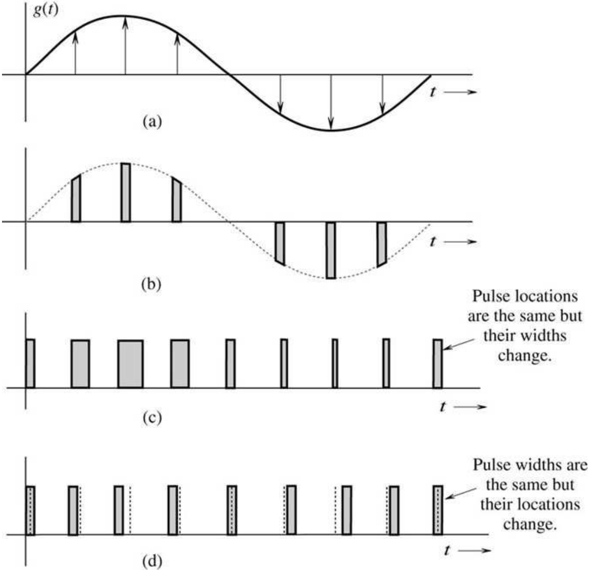
$$E(f) = \begin{cases} 1/P(f)F_0(f) & |f| < B \\ \text{flexible} & B < |f| < 1/T_s - B \\ 0 & |f| > 1/T_s - B \end{cases}$$

Pulse Modulation of Signals

- In many cases, bandwidth of communication link is much greater than signal bandwidth.
- The signal can be transmitted using short pulses with low duty cycle:
  - Pulse amplitude modulation: width fixed, amplitude varies
  - Pulse width modulation: position fixed, width varies
  - Pulse position modulation: width fixed, position varies
- All three methods can be used with time-division multiplexing to carry multiple signals over a single channel



PAM, PWM, PPM: Amplitude, Width, Position



Pulse Amplitude Modulation

- The input to a pulse amplitude modulator is the real-world sample of $g(t)$:

$$g_1(nT_s) = \int_0^{T_s} q(t)g(t - nT_s)\,dt$$

where $q(t)$ is an integrator function. (Width of $q(t)$ should be $\ll T_s$.)

- Each transmitted pulse is narrow with height (or area) proportional to $g_1(nT_s)$. The pulse is integrated to obtain an analog value.

$$\tilde{g}(nT_s) = \int_0^{T_p} q_1(t)g_1(t - nT_s)\,dt$$

where $T_p \ll T_s$

- The original signal $g(t)$ is reconstructed using an equalizer and a low pass filter, as discussed above.

Pulse Width Modulation (PWM)

Pulse width modulation is also called pulse duration modulation (PDM).

PWM is more often used for control than for communication

- Motors
- LEDs: output limunosity is proportional to average current.
- Amplifiers

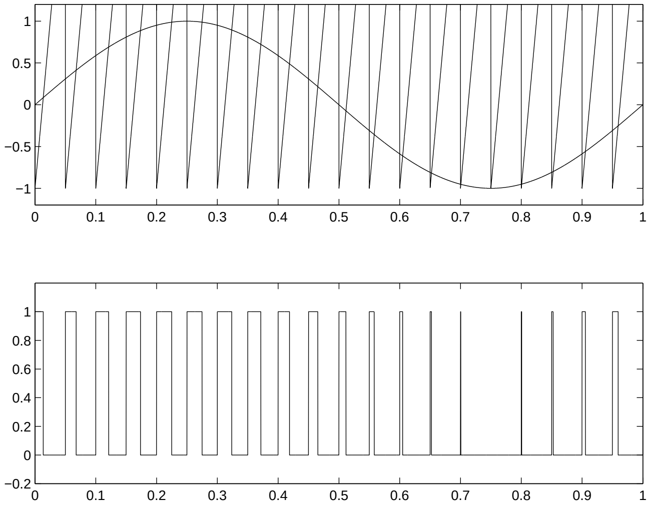
A signal can be recovered exactly from its PWM samples at rate $2B$, provided the bandwidth is $\leq 0.637B$.

J. Huang, K. Padmanabhan, O. M. Collins, *IEEE Trans. Circuits and Systems*, 2011.

PWM (cont.)

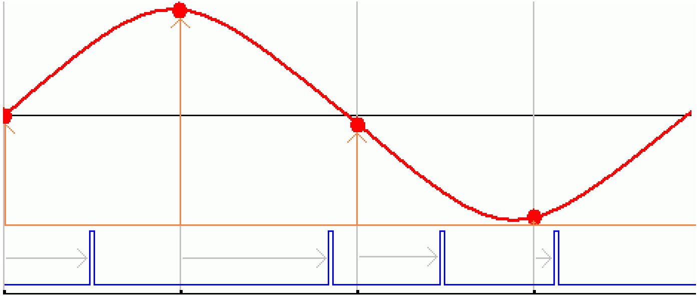
PWM output can be generated by a sawtooth signal gating the input.

Below the pulse width varies from nearly 0 to 1/2 the pulse period.



Pulse Position Modulation (PPM)

The value of the signal determines the delay of the pulse from the clock.



Very common in home automation systems.

Microcontrollers can generate PPM (and PWM) in software. Doesn't require an D/A.

Many Arduinos use PWM to generate analog output waveforms.