Lecture 11: Sampling and Pulse Modulation

John M Pauly

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Sampling and Pulse Trains

- Sampling and interpolation
- Practical interpolation
- Pulse trains
- Analog multiplexing

Based on lecture notes from John Gill

Sampling Theorem

Sampling theorem: a signal $g(t)$ with bandwidth $< B$ can be reconstructed *exactly* from samples taken at any rate $R > 2B$.

Sampling can be achieved mathematically by multiplying by an impulse train. The unit impulse train is defined by

$$\text{III}(t) = \sum_{n=-\infty}^{\infty} \delta(t-k)$$

The unit impulse train is also called the III or comb function.

Sampling a signal $g(t)$ uniformly at intervals T_s yields

$$\overline{g}(t) = g(t)\,\text{III}(t) = \sum_{n=-\infty}^{\infty} g(t)\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t-nT_s)$$

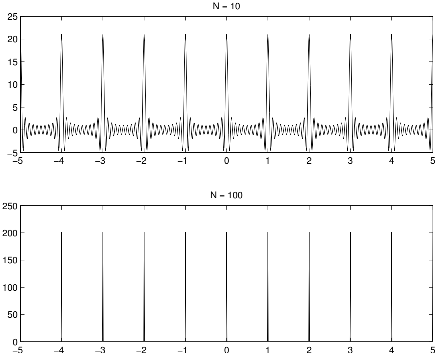
Only information about $g(t)$ at the sample points is retained.

Fourier Transform of $\text{III}(t)$

Fact: the Fourier transform of $\text{III}(t)$ is $\text{III}(f)$.

$$\mathcal{F}\,\text{III}(t) = \sum_{n=-\infty}^{\infty} \mathcal{F}\delta(t-n) = \sum_{n=-\infty}^{\infty} e^{-j2\pi nf} = \sum_{n=-\infty}^{\infty} e^{j2\pi nf} = \text{III}(f)$$

The complex exponentials cancel at noninteger frequencies and add up to an impulse at integer frequencies.



Fourier Transform of Sampled Signal

The impulse train $\text{III}(t/T_s)$ is periodic with period T_s and can be represented as the sum of complex exponentials of all multiples of the fundamental frequency:

$$\text{III}(t/T_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t} \quad (f_s = \frac{1}{T_s})$$

Thus

$$\overline{g} = g(t)\,\text{III}(t/T_s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} g(t) e^{j2\pi n f_s t}$$

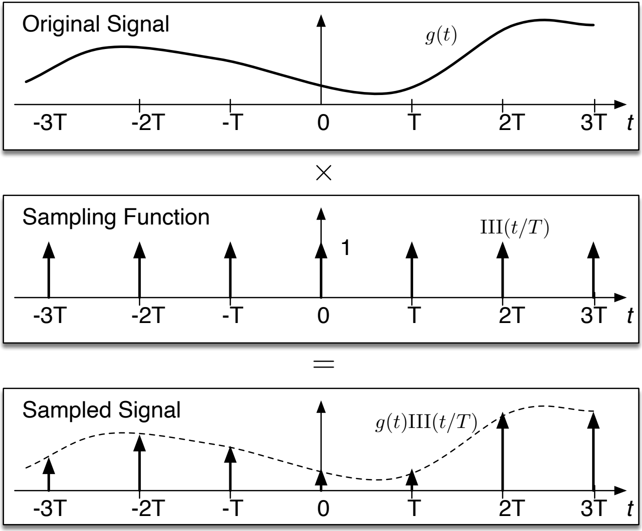
and by the frequency shifting property

$$\overline{G}(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(f - n f_s)$$

This sum of shifts of the spectrum can be written as $\text{III}(f/f_s) * G(f)$.

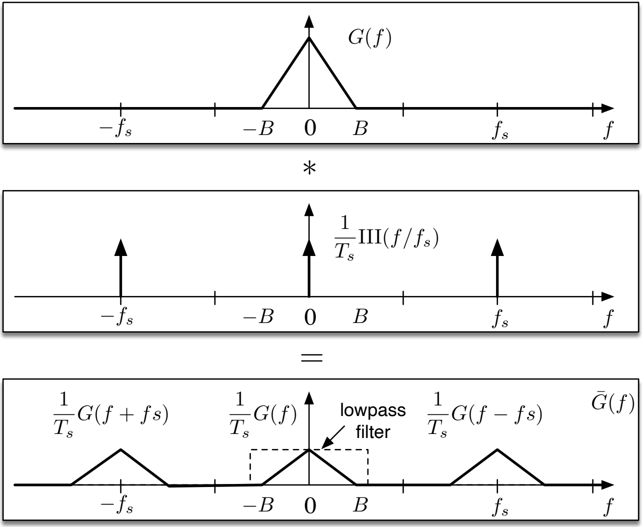
Sampled Signal and Fourier Transform

In the time domain sampling is multiplication by an impulse train



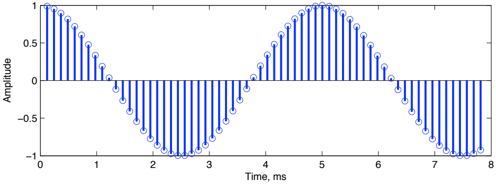
Sampled Signal and Fourier Transform

In the frequency domain sampling is convolution by an impulse train

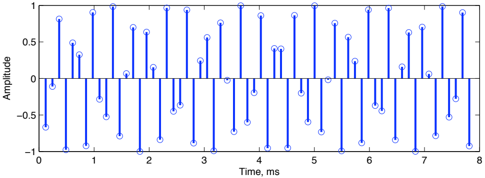


Sampled Cosine Examples

Sometimes it is easy to identify a cosine from its samples

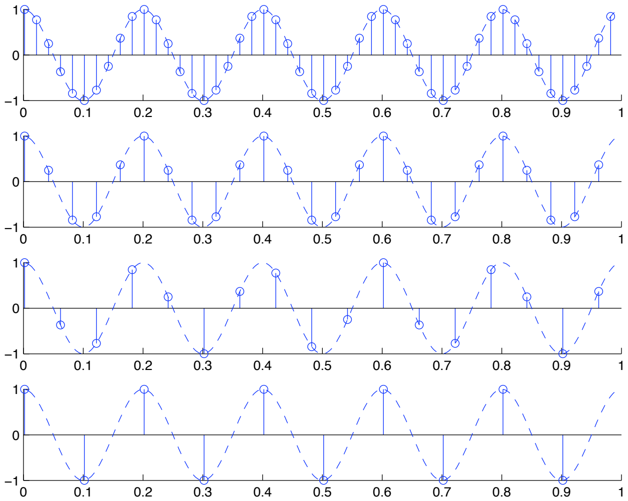


Sometimes it isn't so obvious!



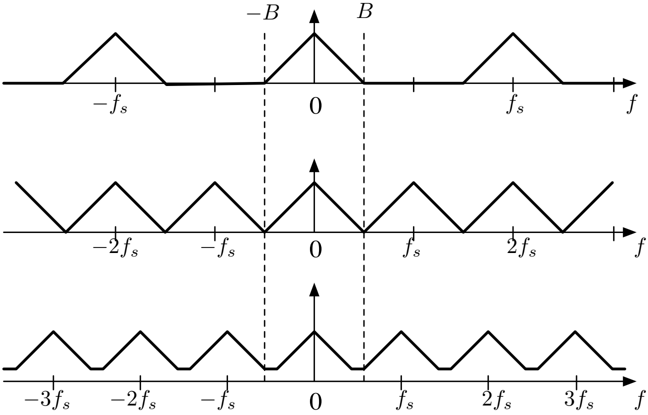
The fact that a signal is bandlimited is a very powerful constraint.

Sampling Examples



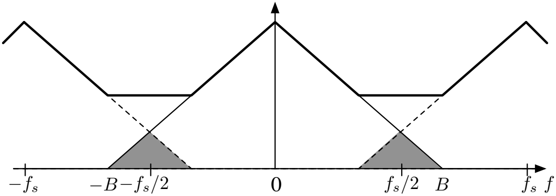
Aliasing and the Minimum Sampling Rate

When the sampling rate is too low, the spectral replicas overlap



This is called aliasing.

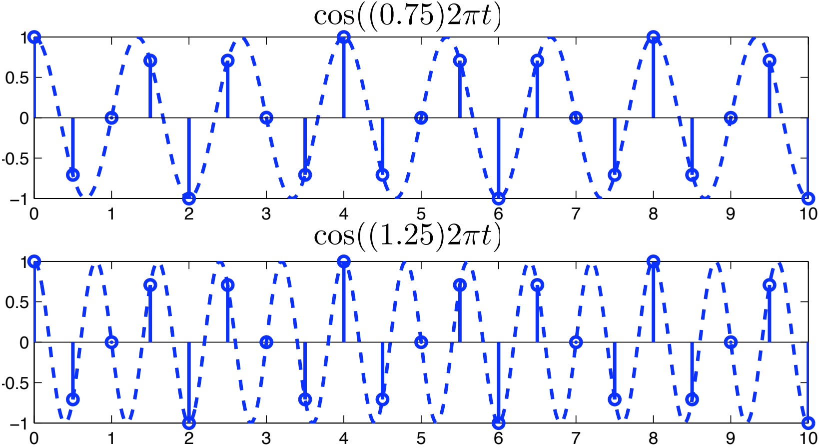
Aliasing



- The shaded frequencies overlap and are ambiguous.
- High positive frequencies wrap around to high negative frequencies
- What signal would you reconstruct if you assumed the signal was actually band limited?

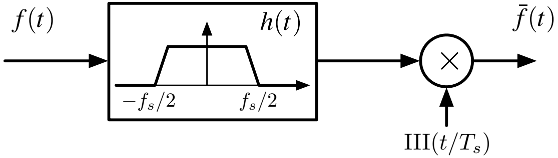
Example of Aliasing

Cosines at frequencies of 0.75 Hz and 1.25 Hz produce exactly the same samples at a sampling rate of 1 Hz



Anti-Aliasing Filter

In practice, a sampler is always preceded by a filter to limit the signal bandwidth to match the sampling rate



This may delete part of the signal if it isn't bandlimited.
It ensures that the signal that is sampled is bandlimited.

Reconstruction from Uniform Samples (Ideal)

If sample rate $1/T_s$ is greater than $2B$, shifted copies of spectrum do not overlap, so low pass filtering recovers original signal.

Cutoff frequency of low pass filter should satisfy

$$B \le f_c \le f_s - B$$

Suppose $f_c = B$. A low pass filter with gain T_s has transfer function and impulse response

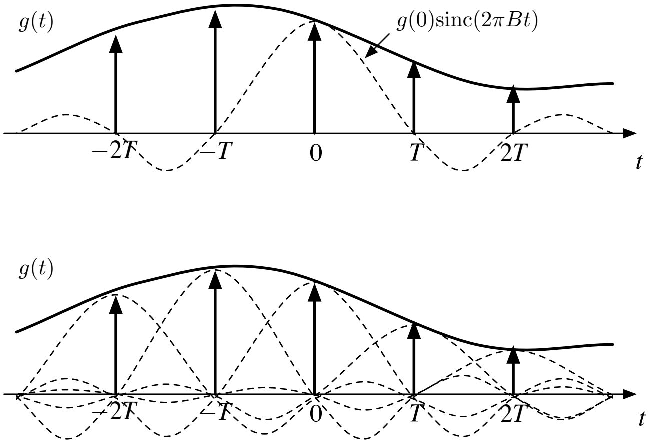
$$H(f) = T_s \Pi\Big(\frac{f}{2B}\Big), \quad h(t) = 2BT_s \,\mathrm{sinc}(2\pi B t)$$

Then if $T_s = 1/2B$

$$\begin{aligned} h(t) * \overline{g}(t) &= \sum_{n=-\infty}^{\infty} h(t) * g(nT_s)\delta(t - nT_s) \\ &= \sum_{n=-\infty}^{\infty} g(nT_s)\,\mathrm{sinc}(2\pi B(t - nT_s)) \end{aligned}$$

Ideal Interpolation

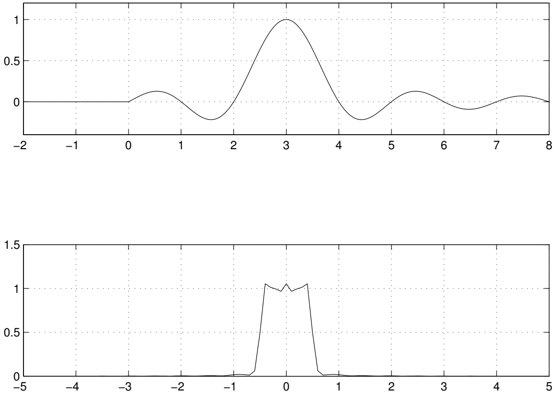
Ideal interpolation represents a signal as sum of shifted sincs.



Practical Interpolation

In practice we require a causal filter. We can delay the impulse response and eliminate values at negative times.

$$\tilde{h}(t) = \begin{cases} h(t - t_0) & t > 0 \\ 0 & t < 0 \end{cases}$$

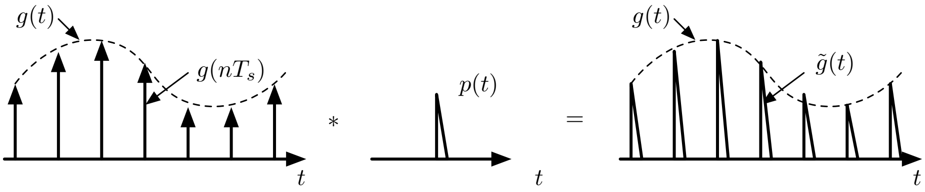


Practical Interpolation (cont.)

In practice, the sampled signal is a sum of pulses, not impulses.

$$\tilde{g}(t) = \sum_{n=-\infty}^{\infty} g(nT_s)p(t - nT_s)$$

$$= p(t) * \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s) = p(t) * \overline{g}(t)$$



Practical Interpolation (cont.)

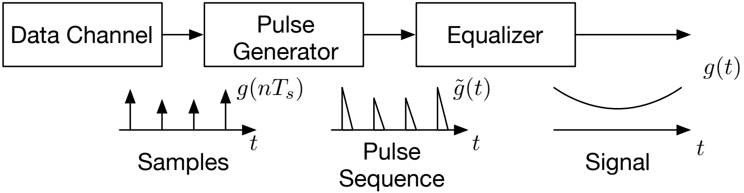
By the convolution theorem,

$$\tilde{G}(f) = P(f) \cdot \frac{1}{T_s} \sum_{n=-\infty}^{\infty} G(f - n f_s)$$

We can recover $G(f)$ from $\tilde{G}(f)$ by low pass filtering to eliminate high frequency shifts and *equalizing* by inverting $P(f)$.

$$E(f) = \begin{cases} T_s / P(f) & |f| < B \\ 0 & |f| > B \end{cases}$$

The transfer function $E(f)$ should not be close to 0 in the pass band.

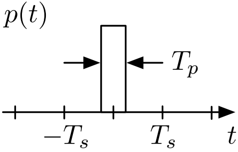


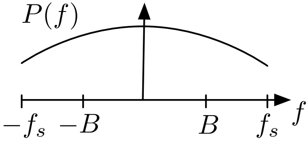
Practical Interpolation (cont.)

Example: rectangular pulses with $T_p < T_s < 1/2B$.

$$p(t) = \Pi\left(\frac{t}{T_p}\right) \;\Rightarrow\; P(f) = T_p \,\mathrm{sinc}(\pi T_p f)$$

This looks like:





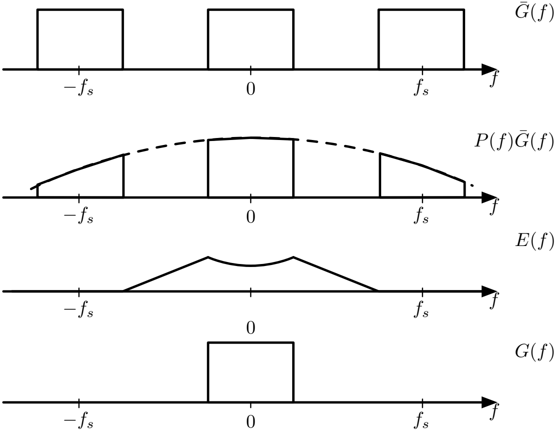
The equalizer should undo the spectral weighting of $P(f)$. The transfer function for the equalizer should satisfy

$$E(f) = \begin{cases} T_s/P(f) & |f| < B \\ \text{don't care} & B < |f| < f_s - B \\ 0 & |f| > f_s - B \end{cases}$$

The last case suppresses all of the spectral replicas.

Practical Interpolation (cont.)

To illustrate, we'll assume that the spectrum $G(f)$ is flat. Then

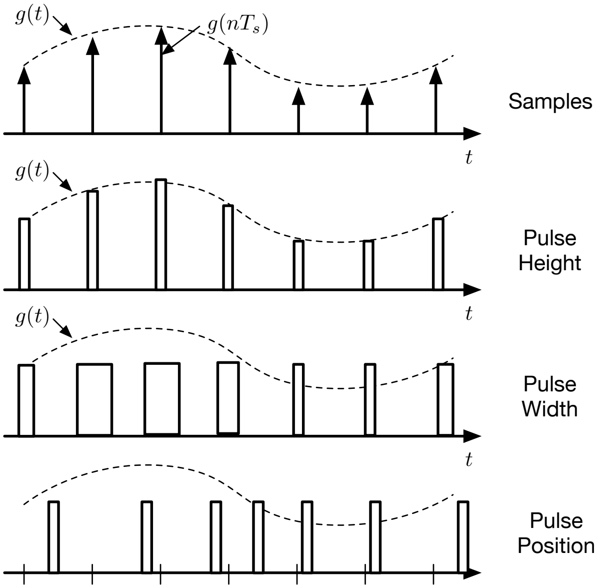


The equalizer $E(f)$ both corrects for the appodization of $P(f)$, and performs the lowpass filtering for optimal interpolation.

Pulse Modulation of Signals

- In many cases, bandwidth of communication link is much greater than signal bandwidth.
- The signal can be transmitted using short pulses with low duty cycle:
  - Pulse amplitude modulation: width fixed, amplitude varies
  - Pulse width modulation: position fixed, width varies
  - Pulse position modulation: width fixed, position varies
- All three methods can be used with time-division multiplexing to carry multiple signals over a single channel

PAM, PWM, PPM: Amplitude, Width, Position



Pulse Amplitude Modulation

- The input to a pulse amplitude modulator is the real-world sample of $g(t)$:

$$g_1(nT_s) = \int_0^{T_s} q(t)g(t - nT_s)\, dt$$

  where $q(t)$ is an integrator function. (Width of $q(t)$ should be $\ll T_s$.)

- Each transmitted pulse is narrow with height (or area) proportional to $g_1(nT_s)$. The pulse is integrated to obtain an analog value.

$$\tilde{g}(nT_s) = \int_0^{T_p} q_1(t)g_1(t - nT_s)\, dt$$

  where $T_p \ll T_s$

- The original signal $g(t)$ is reconstructed using an equalizer and a low pass filter, as discussed above.

Pulse Width Modulation (PWM)

Pulse width modulation is also called pulse duration modulation (PDM).

PWM is more often used for control than for communication

- Motors
- LEDs: output limunosity is proportional to average current.
- Amplifiers

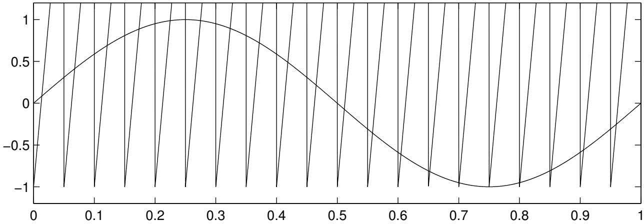
A signal can be recovered exactly from its PWM samples at rate $2B$, provided the bandwidth is $\leq 0.637B$.

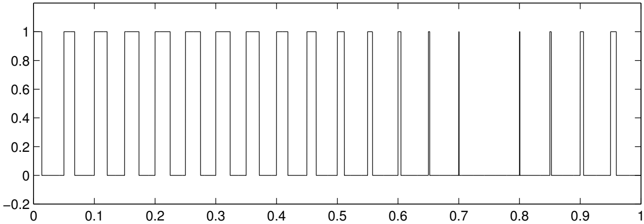
J. Huang, K. Padmanabhan, O. M. Collins, *IEEE Trans. Circuits and Systems*, 2011.

PWM (cont.)

PWM output can be generated by a sawtooth signal gating the input.

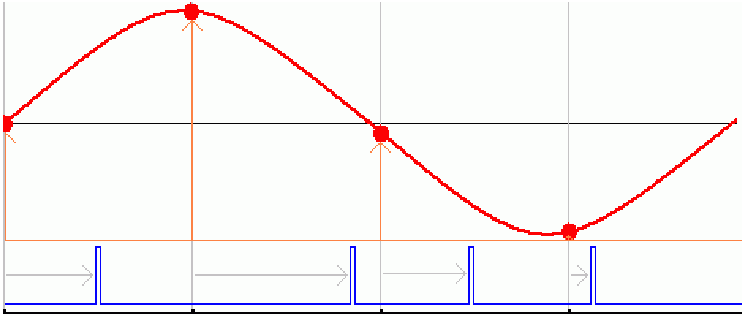
Below the pulse width varies from nearly 0 to 1/2 the pulse period.





Pulse Position Modulation (PPM)

The value of the signal determines the delay of the pulse from the clock.



Very common in home automation systems.

Microcontrollers can generate PPM (and PWM) in software. Doesn't require an D/A.

Many Arduinos use PWM to generate analog output waveforms.

Next Time

- Friday : AM Stereo Transmitter and Receiver
- Monday : Digital pulse trains, pulse code modulation (PCM), quantization
- Wednesday : Digital networks