Problem Set #2 Solutions
Due: Friday April 28, 2017 at 5 PM.

1. We have an input signal \( m(t) \) that is band limited to \( \pm B \) (its full bandwidth is 2B). We want to modulate it to a carrier frequency \( f_c \) with the following system

\[
\begin{align*}
\text{Filter} & \quad m(t) \cos(2\pi f_c t) \\
\cos^3(2\pi f_o t) & \quad \text{Design two different systems with different } f_o \text{'s that will produce the desired output. Specify } f_o \text{ and the filter for each. How large must } f_c \text{ be for this to work (this is seldom a problem!).}
\end{align*}
\]

Solution

First we need to figure out the spectrum of \( \cos^3(2\pi f_0) \). One way to do this is

\[
\cos^3(2\pi f_0 t) = \left( \frac{1}{2} \left( e^{j2\pi f_0 t} + e^{-j2\pi f_0 t} \right) \right)^3
\]

\[
= \frac{1}{8} \left( e^{j2\pi 3f_0 t} + 3e^{j2\pi f_0 t} + 3e^{-j2\pi f_0 t} + e^{-j2\pi 3f_0 t} \right)
\]

\[
= \frac{1}{4} \cos(2\pi 3f_0 t) + \frac{3}{4} \cos(2\pi f_0 t)
\]

Hence there are actually two frequencies we can use, \( f_0 \) and \( 3f_0 \).

If we use \( f_0 = f_c \) we have an additional component at \( 3f_0 \) we should suppress with a lowpass (or bandpass filter). If \( 3f_0 = f_c \), then we have an additional component at \( f_c / 3 \) we should suppress using a highpass filter. These two cases are illustrated below:
This works if the different modulation sidelobes don’t overlap. The limitation in the first case is that $2f_c > 2B$, where $2B$ is the signal bandwidth. In the second case $\frac{2}{3}f_c > 2B$. In practice $f_c >> 2B$, so these are seldom an issue.

2. Another method for generating SSB signals is shown below

where all of the signals are real. The $-\pi/2$ blocks simply take a $\cos()$ and output a $\sin()$ (a Hilbert transform).

(a) Draw a simpler version of this block diagram that uses complex signals. For example, the first two modulators on the left effectively take $m(t)$ and modulate it by $e^{j2\pi(B/2)t}$.

**Solutions**
The way to think about this is that the top branch of the original block diagram is the real part of the signal, and the bottom branch is the imaginary part. The real input signal is multiplied by a complex exponential to produce a complex signal, and the real part goes to the top branch, and the imaginary to the bottom branch. It isn’t quite that simple though, as we will see below.

(b) Assume \( m(t) \) has the spectrum

\[
M(f)
\]

Sketch the spectrum after the first modulator, the lowpass filter.

**Solution**

\[
M(f - B/2)H(f)
\]

The spectrum has been shifted to the right enough to center the lower sideband, which is then isolated with the lowpass filter.

(c) What operation does the last pair of modulators and the sum at the end perform? Sketch the final spectrum.

**Solution**

After the final mixer stage, the lower sideband has been shifted to the carrier frequency.

After the lowpass filters, the original block diagram takes the real part of the signal, and multiplies it by a cosine, and the imaginary part and multiplies it by a sine, and then subtracts the second from the first.

If we look at the complex version from part (a), after the second complex mixer, we take the real part. The output of the complex mixer would require four real mixers, for the real and imaginary components of the signal with each of the real and imaginary components of the complex exponential. The result would be another complex signal.

What the original block diagram does is only compute the real part, since that is the only part we care about. This eliminates two real mixers. Hence the final two real mixers and the difference operation are just computing the real component of the output signal.
The resulting spectrum looks like this:

```
  f_c
  -f_c
```

Note that it has to be Hermitian, since it is real.

This approach is called the *Weaver Method*, and is frequently used in digital implementations. It is convenient in that only modulators and low pass filters are required. No Hilbert transform filter is needed.

3. The modulator described on slides 22 and 23 of Lecture 5 is called a *balanced mixer*. This exploits a non-linearity to modulate a signal to a carrier frequency. In class, we considered the case where there is a quadratic non-linearity. Assume that the nonlinearity also has a cubic term, so

\[ y(t) = ax(t) + bx^2(t) + cx^3(t) \]

Again, assume the inputs are

\[
\begin{align*}
x_1(t) &= \cos(2\pi f_c t) + x(t) \\
x_2(t) &= \cos(2\pi f_c t) - x(t)
\end{align*}
\]

What is the output \( y_1(t) - y_2(t) \) of the balanced mixer? Is this usable as a mixer? What frequencies does it modulate to?

**Solution**

From the notes, if we just had the first two terms (let \( c = 0 \) ) then the result would be

\[ y_1(t) - y_2(t) = 2ax(t) + 4b \cos(2\pi f_c t) \]

If \( c \neq 0 \) there is an additional term

\[
c(x_1^3(t) - x_2^3(t)) = c \left( \cos^3(2\pi f_c t) + 3x(t) \cos^2(2\pi f_c t) + 3x^2(t) \cos(2\pi f_c t) + x^3(t) \right) - c \left( \cos^3(2\pi f_c t) - 3x(t) \cos^2(2\pi f_c t) + 3x^2(t) \cos(2\pi f_c t) - x^3(t) \right)
\]

\[ = 6cx(t) \cos^2(2\pi f_c t) + 2cx^3(t) \]

The result is that with all three terms, we get

\[ y_1(t) - y_2(t) = (2ax(t) + 6cx^3(t)) + 4b \cos(2\pi f_c t) + 6cx(t) \cos^2(2\pi f_c t) \]

There are three frequency components: baseband, \( f_c \), and \( 2f_c \). Since these are separated by \( f_c \), which is large relative to the signal bandwidth \( 2B \), we can filter out the term at \( f_c \) with a bandpass filter, and still use the modulator. We could also use this modulator to generate a term at \( 2f_c \) if we wished.