Signals and Systems: Part 2

- Parsevals theorem for the Fourier Series
- The Fourier transform in $2\pi f$
- Some important Fourier transforms
- Some important Fourier transform theorems
- Convolution and Modulation
- Ideal filters

Based on Notes from John Gill
Exponential Fourier Series: Parseval’s theorem

- Exponential Fourier series:

\[ f(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi f_0 t} \]

where

\[ D_n = \frac{1}{T_0} \int_{0}^{T_0} g(t) e^{-j2\pi f_0 n t} \, dt \]

- Complex exponentials are orthogonal. If \( m \neq n \), and \( T_0 = 1 \),

\[ (e^{j2\pi m t}) \cdot (e^{j2\pi n t}) = \int_{0}^{1} e^{j2\pi (m-n) t} \, dt = 0 \]

since this is the integral of a integer number of cycles of a complex exponential.
Exponential Fourier Series: Parseval’s theorem

It follows that the power of a Fourier series is sum of powers of terms:

\[ \| g(t) \|^2 = \left\| \sum_{n=-\infty}^{\infty} D_n e^{j2\pi nt} \right\|^2 = \sum_{n=-\infty}^{\infty} \| D_n e^{j2\pi nt} \|^2 = \sum_{n=-\infty}^{\infty} \| D_n \|^2 \]

where we have used the Pythagorean theorem again, since the complex exponentials are orthogonal signals.

We can calculate power in either time or frequency domain.

Also called Parseval’s identity (1799), Plancherel’s theorem (1910) and Rayleigh’s theorem.
In EE 102A, the Fourier transform was

\[ G(j\omega) = \int_{-\infty}^{\infty} g(t)e^{-j\omega t} dt \]

and the inverse transform was

\[ g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega)e^{j\omega t} d\omega \]

We used \( j\omega \) to make all of the transforms to be similar in form (Fourier, Laplace, DTFT, \( z \))
In this class we will use $2\pi f$, instead of $\omega$. If we replace $\omega$ with $2\pi f$, the Fourier transform is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} \, dt.$$  

Noting that $d\omega = 2\pi \, df$, the inverse transform is then

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} \, df.$$  

Note that these are almost completely symmetric. Only the sign of the complex exponential changes.

This will simplify a lot of the transforms and theorems.
Fourier Transform Existence

Fact: if $g(t)$ is absolutely integrable, i.e.,
\[ \int_{-\infty}^{\infty} |g(t)| \, dt < \infty \]
then $G(f)$ exists for every frequency $f$ and is continuous.

Fact: if $g(t)$ has finite energy, i.e.,
\[ \int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty \]
then $G(f)$ exists for “most” frequencies $f$ and has finite energy.

Fact: if $g(t)$ is periodic and has a Fourier series, then
\[ G(f) = \sum_{n=-\infty}^{\infty} G(nf_0) \delta(f - nf_0) \]
is a weighted sum of impulses in frequency domain.
Fourier Transform Examples

One-sided exponential decay is defined by $e^{-at}u(t)$ with $a > 0$:

$$g(t) = \begin{cases} 
0 & t < 0 \\
e^{-at} & t > 0 
\end{cases}$$

The Fourier transform of one-sided decay is (simple calculus):

$$G(f) = \int_0^\infty e^{-j2\pi ft} e^{-at} \, dt = \int_0^\infty e^{-a+j2\pi ft} \, dt$$

$$= \left[ \frac{e^{-(a+j2\pi ft)}}{-a+j2\pi f} \right]_{t=0}^{t=\infty} = \frac{1}{a+j2\pi f}$$

Since $g(t)$ has finite area, its transform is continuous.

$g(t)$ is real but its transform is complex valued. This is the usual situation.
Fourier Transform Examples (cont.)

We can rationalize $G(f)$.

$$G(f) = \frac{1}{a + j2\pi f} = \frac{a - j2\pi f}{a^2 + 4\pi^2 f^2} = \frac{a}{a^2 + 4\pi^2 f^2} - j\frac{2\pi f}{a^2 + 4\pi^2 f^2}$$

We can better picture $G(f)$ using polar representation.

$$|G(f)| = \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}}, \quad \theta_{G}(f) = \angle G = -\tan^{-1}\left(\frac{-2\pi f}{a}\right)$$
Fourier Transform Examples (cont.)

Fourier transform at $f = 0$:

$$G(0) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi \cdot 0 \cdot t} \, dt = \int_{-\infty}^{\infty} g(t) \, dt$$

is the area under $g(t)$, called the DC value.

For $g(t) = e^{-at}u(t)$, $G(0) = \frac{1}{a}$ is the only real value and largest in magnitude.

For one-sided decay, $G(-f) = G^*(f)$, complex conjugate of $G(f)$. This is true for all real-valued signals.

$$G^*(f) = \left( \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft} \, dt \right)^* = \int_{-\infty}^{\infty} g(t)^* (e^{-j2\pi ft})^* \, dt$$

$$= \int_{-\infty}^{\infty} g(t)e^{j2\pi ft} \, dt = \int_{-\infty}^{\infty} g(t)e^{-j2\pi(-f)t} \, dt = G(-f)$$

We need only positive frequencies for real-valued signals.
Duality

Fourier inversion theorem: if \( g(t) \) has a Fourier transform \( G(f) \),

\[
G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi ft} \, dt
\]

\[
g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi ft} \, df
\]

The inverse transform differs from the forward transform only in the sign of the exponent.

If we consider \( G(f) \) to be a function of \( t \) instead of \( f \), and apply the Fourier Transform again

\[
\int_{-\infty}^{\infty} G(t) e^{-j2\pi ft} \, dt = \int_{-\infty}^{\infty} G(t) e^{j2\pi (-f)t} \, dt = g(-f)
\]

We can summarize this as

\[
\mathcal{F}\{g(t)\} = G(f) \Rightarrow \mathcal{F}\{G(t)\} = g(-f)
\]

This is the principle of duality.
Important Fourier Transforms

The unit rectangle function $\Pi(t)$ is defined by

$$\Pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & |t| > \frac{1}{2} \end{cases}$$

Its Fourier transform is

$$\mathcal{F}\{\Pi(t)\} = \int_{-\infty}^{\infty} e^{-i2\pi ft} \Pi(t) \, dt = \int_{-1/2}^{1/2} e^{-i2\pi ft} \, dt = \frac{\sin \pi f}{\pi f} = \text{sinc}(\pi f)$$

Fact: every finite width pulse has a transform with unbounded frequencies.
By duality, we can also find

\[ \text{sinc}(\pi t) \Leftrightarrow \Pi(f) \]

since \( \Pi(f) \) is even, so \( \Pi(f) = \Pi(-f) \).
Fourier Transform Time Scaling

If $a > 0$ and $g(t)$ is a signal with Fourier transform $G(f)$, then

$$\mathcal{F} \{g(at)\} = \int_{-\infty}^{\infty} g(at)e^{-j2\pi ft} dt = \int_{-\infty}^{\infty} g(u)e^{-j2\pi fu/a} (du/a) = \frac{1}{a} G\left(\frac{f}{a}\right)$$

If $a < 0$, change of variables requires reversing limits of integration:

$$\mathcal{F} \{g(at)\} = -\frac{1}{a} G\left(\frac{f}{a}\right)$$

Combining both cases:

$$\mathcal{F} \{g(at)\} = \frac{1}{|a|} G\left(\frac{f}{a}\right)$$

Special case: $a = -1$. The Fourier transform of $g(-t)$ is $G(-f)$.

Compressing in time corresponds to expansion in frequency (and reduction in amplitude) and vice versa.

The sharper the pulse the wider the spectrum.
Fourier Transform Time Scaling Example

The transform of a narrow rectangular pulse of area 1 is

\[
\mathcal{F}\left\{ \frac{1}{\tau} \Pi(t/\tau) \right\} = \text{sinc}(\pi \tau f)
\]

In the limit, the pulse is the unit impulse, and its transform is the constant 1.

We can find the Fourier transform directly:

\[
\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} \, dt = e^{-j2\pi ft}\bigg|_{t=0} = 1
\]

so

\[ \delta(t) \Leftrightarrow 1 \]

The impulse is the mathematical abstraction of a signal whose Fourier transform has magnitude 1 and phase 0 for all frequencies.

By duality, \(\mathcal{F}\{1\} = \delta(f)\). All DC, no oscillation.

\[ 1 \Leftrightarrow \delta(f) \]
Important Fourier Transforms (cont.)

- Shifted impulse $\delta(t - t_0)$:

$$\mathcal{F}\{\delta(t - t_0)\} = \int_{-\infty}^{\infty} \delta(t - t_0)e^{-j2\pi ft} \, dt = e^{-j2\pi ft_0}$$

This is a complex exponential in frequency.

$$\delta(t - t_0) \iff e^{-j2\pi ft_0}$$

Then by duality

$$e^{j2\pi f_0 t} \iff \delta(f - f_0)$$

- Sinuoids:

$$\mathcal{F}\{\cos 2\pi f_0 t\} = \mathcal{F}\left\{\frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\right\} = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$$

$$\mathcal{F}\{\sin 2\pi f_0 t\} = \mathcal{F}\left\{\frac{1}{2i}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t})\right\} = \frac{1}{2i}\delta(f - f_0) - \frac{1}{2i}\delta(f + f_0)$$
**Important Fourier Transforms (cont.)**

- **Laplacian pulse** \( g(t) = e^{-a|t|} \) where \( a > 0 \). Since

\[
g(t) = e^{-at}u(t) + e^{at}u(-t),
\]

we can use reversal and additivity:

\[
G(f) = \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + 4\pi^2 f^2}.
\]

This is twice the real part of the Fourier transform of \( e^{-at}u(t) \).

- **The signum function can be approximated as**

\[
\text{sgn}(t) = \lim_{a \to 0} \left( e^{-at}u(t) - e^{at}u(-t) \right)
\]
This has the Fourier transform

\[ \mathcal{F}\{\text{sgn}(t)\} = \lim_{a \to 0} \left( \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f} \right) = \frac{1}{j\pi f} \]

The unit step function is

\[ u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t) \]

This has the Fourier transform

\[ \mathcal{F}\{u(t)\} = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} \]
Fourier Transform Properties

- Time delay causes linear phase shift.

\[ \mathcal{F}\{g(t - t_0)\} = \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft} \, dt = \int_{-\infty}^{\infty} g(u)e^{-j2\pi f(u+t_0)} \, du \]

\[ = e^{-j2\pi ft_0} \int_{-\infty}^{\infty} g(u)e^{-j2\pi fu} \, du = e^{-j2\pi ft_0} G(f) \]

So,

\[ g(t - t_0) \Leftrightarrow e^{-j2\pi ft_0} G(f) \]

- Then, by duality, we get frequency shifting (modulation):

\[ e^{j2\pi f_c t} g(t) \Leftrightarrow G(f - f_c) \]

\[ \cos(2\pi f_c t) g(t) \Leftrightarrow \frac{1}{2} G(f + f_c) + \frac{1}{2} G(f - f_c) \]
Fourier Transform Properties (cont.)

- Convolution in time. The convolution of two signals is

\[ g_1(t) \ast g_2(t) = \int_{-\infty}^{\infty} g_1(u)g_2(t-u) \, du \]

- The Fourier transform of the convolution is the product of the transforms.

\[ g_1(t) \ast g_2(t) \iff G_1(f)G_2(f) \]

The Fourier transform reduces convolution to a simpler operation.

- Note that there is no \( \frac{1}{2\pi} \) factor for frequency domain convolution, as there was in EE 102A.
Multiplication in time.

\[ g_1(t)g_2(t) \equiv \int_{-\infty}^{\infty} G_1(\lambda)G_2(f - \lambda) \, d\lambda \]

Again, there is no \( \frac{1}{2\pi} \) factor, as there was in EE 102A.

Convolution in one domain goes exactly to multiplication in the other domain, and multiplication to convolution.

The modulation theorem is a special case.

\[
\mathcal{F}\{g(t) \cos(2\pi f_c t)\} = \mathcal{F}\{g(t)\} \ast \mathcal{F}\{\cos(2\pi f_c t)\} \\
= G(f) \ast \left( \frac{1}{2} \delta(f + f_0) + \frac{1}{2} \delta(f - f_0) \right) \\
= \frac{1}{2} G(f + f_c) + \frac{1}{2} G(f - f_c)
\]
The triangle function $\Delta(t)$ and its Fourier transform

- The book defines the triangle function $\Delta(t)$ as

$$\Delta(t) = \begin{cases} 1 - 2|x| & |x| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

This is another unfortunate choice, but not as bad as $\text{sinc}(t)$!

- The triangle function can be written as twice the convolution of two rectangle functions of width $\frac{1}{2}$.

$$\Delta(t) = 2 \, \Pi(2t) \ast \Pi(2t)$$

where the factor of 2 is needed to make the convolution 1 at $t = 0$. 
The Fourier transform is then

\[
\mathcal{F}\{\Delta(t)\} = \mathcal{F}\{2\Pi(2t) * \Pi(2t)\} \\
= 2\mathcal{F}\{\Pi(2t)\}^2 \\
= 2 \left( \frac{1}{2} \text{sinc} \left( \frac{\pi}{2} f \right) \right)^2 \\
= \frac{1}{2} \text{sinc}^2 \left( \frac{\pi}{2} f \right)
\]

Then

\[
\Delta(t) \Leftrightarrow \frac{1}{2} \text{sinc}^2 \left( \frac{\pi}{2} f \right)
\]
Modulation Theorem

Modulation of a baseband signal creates replicas at ± the modulation frequency.
Applications of Modulation

- For transmission by radio, antenna size is proportional to wavelength. Low frequency signals (voice, music) must be converted to higher frequency.

- To share bandwidth, signals are modulated by different carrier frequencies.
  - North America AM radio band: 535–1605 KHz (10 KHz bands)
  - North America FM radio band: 88–108 MHz (200 KHz bands)

  Frequencies can be reused in different geographical areas.

  With digital TV, channel numbers do not correspond to frequencies.

Rats laughing: [http://www.youtube.com/watch?v=j-admRGFVNM](http://www.youtube.com/watch?v=j-admRGFVNM)
Bandpass Signals

Bandlimited signal: \( G(f) = 0 \) if \(|f| > B\).

Every sinusoid \( \sin(2\pi f_c t) \) has bandwidth \( f_c \).

If \( g_c(t) \) and \( g_s(t) \) are bandlimited, then

\[
m(t) = g_s(t) \cos(2\pi f_c t) + g_s(t) \sin(2\pi f_c t)
\]

is a bandpass signal. Its Fourier transform or spectrum is restricted to

\[
f_c - B < |f| < f_c + B
\]

The bandwidth is \((f_c + B) - (f_c - B) = 2B\).

Most signals of interest in communications will be either bandpass (RF), or baseband (ethernet).
Filters

A filter is a system that modifies an input. (E.g., an optical filter blocks certain frequencies of light.)

In communication theory, we usually consider linear filters, which are linear time-invariant systems.

\[ L(v_1(t) + v_2(t)) = L(v_1(t)) + L(v_2(t)), \quad L(av(t)) = aL(v(t)) \]

Fundamental fact: every LTIS is defined by convolution:

\[ Lv(t) = h(t) \ast v(t) = \int_{-\infty}^{\infty} h(u)v(t-u)\,du = \int_{-\infty}^{\infty} h(t-u)v(u)\,du \]

The signal \( h(t) \) is called the impulse response because

\[ L\delta(t) = \int_{-\infty}^{\infty} \delta(u)h(t-u)\,du = h(t) \]

If a system is not time invariant, the impulse response is \( h(t_1, t_2) \); the shape of the output for input \( \delta(t - t_1) \) depends on \( t_1 \).
Transfer Function

By the convolution theorem,

\[ h(t) \ast v(t) \iff H(f)V(f) \]

\( H(f) \) is called the *transfer function*.

Many systems are best understood in the frequency domain.

Example: low pass filter with cutoff frequency \( B \):

\[ H(f) = \Pi(f/2B)e^{-j2\pi ft_d} \iff 2B \text{sinc}(2\pi B(t - t_d)) \]

Note that the system is not (cannot) be causal.
Low Pass Filter Example

\[ x(t) = \cos(2\pi(0.3)t) + \cos(2\pi(0.8)t), \quad h(t) = \text{sinc}(\pi t), \quad y(t) = \cos(2\pi(0.3)t) \]
Butterworth Filter: Nonideal Low-Pass Filter

Butterworth filter has

$$|H(f)| = \frac{1}{\sqrt{(f/B)^{2n}}}$$
Butterworth Filter vs. Ideal Lowpass Filter
High Pass and Band Pass Filters

If the phase is zero

\[ h_{\text{highpass}}(t) = \delta(t) - 2B \text{sinc}(2\pi Bt) \]

\[ h_{\text{bandpass}}(t) = 2 \cos(2\pi f_0 t) 2B \text{sinc}(2\pi Bt) \]

In practice there will be an additional linear phase in the spectral profile, corresponding to a delay in time. This makes the filters realizable.
Band Pass Filter Example

\[ x(t) = \cos(2\pi(0.3)t) + \cos(2\pi(0.8)t), \ h(t) = \cos 2\pi t \ \text{sinc} \ \pi t, \]
\[ y(t) = \cos(2\pi(0.8)t) \]
Low Pass Filter: $x(t) = \Pi\left(\frac{t}{4}\right)$, $h(t) = \text{sinc } t$
Low Pass Filter: $x(t) = \Pi\left(\frac{t}{4}\right)$, $h(t) = \text{sinc} \ t$, $y(t) = h(t) \ast x(t)$. 
Band Pass Filter: $h(t) = \cos 2\pi t \ \text{sinc} \ t$, $x(t) = \Pi\left(\frac{t}{4}\right)$, 
$y(t) = h(t) \ast x(t)$
Next time

- Lab this Friday: How do you figure out where signals are? Spectrograms and waterfall plots.
- Next class Monday: 3.6 – 3.8 in Lathi and Ding. Signal distortion, power spectral density, correlation and autocorrelation.
- Wednesday: Begin Chapter 4. Analog modulation schemes.
- Lab next Friday: finding and decoding airband AM.