Problem Set #8

Due Date: Wednesday, December 9, 2005. Submit in class.

Problems:

1. Rayleigh Waves [30 points]
   (a) Find the six roots of the Rayleigh equation for a perfectly compressible \((\mu \text{ finite, } \lambda = 0)\)
   semi-infinite isotropic solid with a vacuum interface. Hint: one of the roots is \(\sqrt{2}\). [10 points]
   (b) Which of the six roots is the \(+x\)-directed slow-wave solution? Find the Rayleigh wave speed. [5 points]
   (c) Sketch the particle motion at the surface and find the depth where the particle motion is
   only in the \(z\) direction. [15 points]

   For a wave in a stratified and isothermal atmosphere, assume a solution having vertical displacement
   \[ \zeta = \zeta_0 e^{ix} \cos(\omega t - ax - \gamma z) \]
   where \((2\nu)^{-1}\) is the scale height of the atmosphere. Consider the incompressible case \((c \to \infty)\):
   (a) Find the expression for \(\xi\). [4 points]
   (b) For a fixed point in space \((x = 0, z = 0)\), eliminate time from the expressions for \(\xi\) and \(\zeta\)
   to obtain the following equation for the particle orbits [10 points]:
   \[ \xi^2 + \zeta^2 \left( \frac{N^2}{\omega^2} - 1 \right) + 2 \frac{\nu}{\alpha} \xi \zeta = \frac{\nu^2}{\alpha^2} \zeta_0^2 \]
   (c) Describe the shape of the orbit for the cases below. For v through viii, compare the
   lengths of the vertical and horizontal displacements for the prescribed ellipses. Use the
   orbit equation above. [16 points]
   i. \(\omega < N\)
3. Equatorial waves [30 points]

When we apply the shallow water equations on the rotating earth in Cartesian geometry, we obtain the following characteristic equation for propagating waves near the equator

\[
\frac{\sqrt{gh}}{\beta} \left( \frac{\omega^2}{gh} - k^2 - \frac{k}{\omega^2} \right) = 2n + 1
\]

where \( h \) is the equivalent depth of the ocean near the equator (constant), \( \beta \) is the Coriolis parameter (constant near the equator), \( g \) is the acceleration due to gravity (constant), \( k \) is the longitudinal wavenumber, \( \omega \) is the angular frequency, and \( n \) (an integer) corresponds to the meridional mode number (for different solutions).

(a) Plot the \( \omega \) vs \( k \) diagrams for each of the following modes.

(b) Comment on the direction of propagation (positive \( k \) corresponds to eastward propagating waves along the equator) and the nature of dispersion.

(c) What are the corresponding cut-off frequencies for eastward propagating waves (positive \( k \))?

i. Mixed Rossby-gravity wave (\( n = 0 \))
ii. Kelvin wave (\( n = -1 \))
iii. Rossby wave (\( n = 2 \), Low frequency)
iv. Inertio-gravity wave (\( n = 2 \), High frequency)

Note: You can plot \( \omega' = \omega/(\beta \sqrt{gh})^{\frac{1}{2}} \) vs \( k' = k(\sqrt{gh}/\beta)^{\frac{1}{2}} \) to make your plots independent of the constants, \( g, h \) and \( \beta \).