Calculation of Driving Point Z

Fraction - 1 Dipole with Symmetrical I(z)

\[ J(\vec{r}) + j\omega t \]

\[ J(\vec{r}) e^{j\omega t} \] will be current density distribution on an antenna with an idealized source generator.

Some preliminaries:
\[ \vec{H} = \nabla \times \vec{A}, \quad \nabla \times \vec{H} = \nabla \times \nabla \times \vec{A} \]

\[ \nabla \times \vec{H} = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \vec{J} + j \omega \varepsilon \vec{E} \]

but \[ \nabla^2 \vec{A} + k^2 \vec{A} = -\frac{\vec{J}}{\mu} \Rightarrow \nabla^2 \vec{A} = -\frac{\vec{J}}{\mu} - k^2 \vec{A} \]

eliminate \( \vec{J} \) from above to obtain

\[ \nabla (\nabla \cdot \vec{A}) + k^2 \vec{A} = j \omega \varepsilon \mu \vec{E} \]

which relates \( \vec{A}, \vec{E} \), and is useful in problems where \( \vec{E} \) is the specified B.C.
But still
\[ \mathbf{A} = \frac{j \kappa R}{4\pi} \int_{\text{vol}} \frac{\mathbf{J} (\mathbf{r'}) e d\mathbf{r'}}{R}, \quad \mathbf{R} = \mathbf{r} - \mathbf{r'}, \]
as before.

Combining the reation integral and the last equation, above, we obtain \textbf{Pocklington's Equation}

\[ (\nabla^2 + k^2) \int_{\text{vol}} \mu J (\mathbf{r'}) e d\mathbf{r'} = j \omega e \mathbf{E} \]

Diff. Eq.

\[ \frac{\partial \mathbf{A}}{\partial t}, \]

\[ \text{integral Eq. \hspace{1cm}} \]

\[ \nabla (\nabla \cdot \mathbf{A}) \text{ differentiated w.r.t } \mathbf{F}, \text{ not } \mathbf{F'}, \]

will represent impressed field due to sources.
With regard to the differentiation implied by

" \|D_F \|_{\infty} ^{\|D_F \|_{\infty}} " \n
Think of \( R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \),

i.e., \( R = R(x,x'; y,y'; z,z') \)

Primed quantities represent source points,
unprimed quantities represent observation points.
The "unprimed R" associated with \( D_F \)
means differentiate w.r.t. the observation
point.
Strength and Utility of Pocklington's Eq. for Antennas

1. If $\mathbf{E}(\mathbf{r})$ is known at all points occupied by an antenna, then P. Eq. is an integral equation for the unknown current distribution $\mathbf{F}(\mathbf{r})$.

Most antennas are composed of good conductors, so usually $\mathbf{E}(\mathbf{r}) \neq \mathbf{0}$, except in the vicinity of the source or driving transmission line, called the feed point.

* or knowing $\mathbf{F}(\mathbf{r})$, one can get $\mathbf{E}$, see below.
2. If a good estimate of $E(\mathbf{r})$ is available, then numerical techniques will provide the solution for the current distribution.

3. Knowing $\mathbf{F}(\mathbf{r})$ one can obtain the driving point $Z$ and the radiation pattern.

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N.B. Previously, $F$ and $F'$ have referred to different regions of space. Here they will typically refer to or explore the same region of space.
How to apply this to Dipoles?

Hollow tube - current flows on outer walls. Current is linear, z-directed, on surface from \(-\frac{L}{2} \leftrightarrow -\frac{L}{2}\) and \(\frac{L}{2} \leftrightarrow \frac{L}{2}\).

\(K_a \ll 1\) so fields inside the tube and deviations from circular symmetry can be neglected.

Generator applies cylindrical \(E\) at the origin, \(\frac{E_z}{z} = 0\) elsewhere.

For our purposes, assume circular cross-section \(u/\) radius \(a\).
$V$ is not a function of $\alpha, \phi$.

Voltage is positive

$E$ is directed downward, towards negative $z$. 
As there are only $z$-directed fields present, presumably driving $z$-directed currents, P. Eq., above, becomes...

\[
\begin{aligned}
(\dfrac{\partial^2}{\partial z^2} + k^2) 
\left( \int \frac{K_0(\phi', \phi)}{4\pi R} e^{-jkR} a \, d\phi' \, dz' \right) &= j\omega \epsilon_\infty E_z(\phi, z) \\
\text{Surface} &= \text{Hollow generator}
\end{aligned}
\]

$E_z \neq 0$, except at the terminals. \( V = - \int_{-\delta/2}^{\delta/2} E_z(a, z) \, dz = 1 \)
With \( J \) vanishing, \( E_3(\Phi,z) \) takes on the characteristics of \( \delta(z) \) - a Dirac Delta.

\[
\lim_{\delta \to 0} \int_{-\delta/2}^{\delta/2} E_3(z) \, dz = -1 \text{ (volts)} \Rightarrow E_3(z) = 0, \ z \neq 0.
\]

1. Eq. above is a general relationship between \( \tilde{F} \) and \( E \). But how to solve it? We will apply diff. eq. for \( A_3(\zeta) \) to the dipole, since we know \( E_3 \) there, under the assumption that \( \delta \to 0 \).
Here we have pictured the element as a right-circular cylinder. But this is not an important feature. Analysis can easily be carried out for any cylinder; other common cross sections might be

- Hollow or solid
- Square
- Rectangular (called strip of very thin)
- Bundle of wires, twisted or not twisted
- Equivalent circular section exist for all of these.
- Parallel groups of conductors.
(An Aside ...)  

Clarification of Pocklington's Equation

Why does only \( \frac{\partial^2}{\partial z^2} \) appear on LHS?

This can be seen by plugging through...

\[
\nabla \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r \vec{A}_r) + \frac{1}{r} \frac{\partial \vec{A}_\theta}{\partial \theta} + \frac{\partial \vec{A}_z}{\partial z} \\
\n\nabla \times \vec{A} = \frac{\partial}{\partial r} \vec{A}_\theta - \frac{1}{r} \frac{\partial}{\partial \theta} \vec{A}_r + \frac{\partial}{\partial z} \vec{A}_\phi
\]

\[
\partial f = \frac{\partial f}{\partial r} \vec{A}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{A}_\theta + \frac{\partial f}{\partial z} \vec{A}_z
\]
\[ \nabla (\nabla \cdot \mathbf{A}) = \nabla \left( \frac{1}{r} \frac{\partial}{\partial r} (rA_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_3}{\partial z} \right) \]

\[ = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (rA_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_3}{\partial z} \right) \hat{r} \]

\[ + \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial}{\partial r} (rA_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_3}{\partial z} \right) \hat{\phi} \]

\[ + \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial}{\partial r} (rA_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_3}{\partial z} \right) \hat{z} \]
We are trying to solve the vector equation (a)

\[ \nabla (\nabla \cdot \vec{A}) + k^2 \vec{A} = j \omega \vec{E} \]

\[ \frac{\partial}{\partial r} A_{2r} + k^2 A_r = j \omega E_r \quad (\hat{u}_r) \]

\[ \frac{1}{r} \frac{\partial}{\partial \phi} A_\phi + k^2 A_\phi = j \omega E_\phi \quad (\hat{u}_\phi) \]

\[ \frac{\partial^2}{\partial z^2} A_z + k^2 A_z = j \omega E_z \quad (\hat{u}_z) \]
\[ \nabla (\nabla \cdot \mathbf{A}) = ? \]

\[ \hat{\mathbf{A}} = \hat{u}_3 \int_0^{2\pi} \int_0^{\pi/2} \frac{K_2(\phi', \mathbf{r}')}{4\pi R} e^{-jkR} \, a \, d\phi' \, d_3' \]

\[ \nabla (\nabla \cdot \mathbf{A}) = \frac{\partial}{\partial x^3} A_3 \, \hat{u}_r + \frac{1}{r} \frac{\partial}{\partial \phi} A_3 \, \hat{u}_\phi + \frac{\partial^2}{\partial z^2} A_3 \, \hat{u}_z \]
But note that since we have z-directed currents only

\[ A_r = A_\phi = 0 \]

While Pocklington’s Equation \( \rightarrow A_z \)

given \( \bar{A} = 0 \cdot \hat{u}_r + 0 \cdot \hat{u}_\phi + A_z \hat{u}_z \)

\[ E_r = \frac{1}{j \mu \varepsilon} \frac{\partial}{\partial z} A_z \]

\[ E_\phi = \frac{1}{j \mu \varepsilon} \frac{\partial}{\partial \phi} A_z \]
Note that if only \( \delta \)-directed currents are present, then \( \mathbf{A} = A_\delta \mathbf{a}_\delta \), and

\[
\mathbf{j}_W \mathbf{E}_\delta = \frac{\partial^2 A_\delta}{\partial z^2} + k^2 A_\delta
\]

so specifying \( E_\delta \) is sufficient to determine \( A_\delta, \overline{\mathbf{j}} \)
(see above)

This does not imply that other components of \( \mathbf{E} \) are necessarily \( = 0 \), only that they are not required to get \( \overline{\mathbf{j}} \).

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*Note that \( \mathbf{j} \) is to be equated to \( A_\delta \) (End Aside)
That is, we will obtain the solution in two parts.

1. Solve the diff. eq. for $\bar{A}_z$.

2. Solve the radiation integral for $\bar{F} = C_3$.

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* Since the diff. eq. doesn't involve cross terms of vector components and because our boundary condition will be expressed in terms of $E_z$, we can solve for $A_z$ independently of $A_x, A_y$. 
Solution:

\[ A_3(z) = \xi_1 \cos k z + \xi_2 \sin k z \quad z > 0 \]

\[ = \xi_3 \cos k z + \xi_4 \sin k z \quad z < 0 \]

is a solution to the homogeneous wave equation

\[ \left( \frac{\partial^2}{\partial z^2} + k^2 \right) A_3(z) = 0, \] which is OK except

at \( z = 0 \).

At \( z = 0 \)

\[ \left( \frac{\partial^2}{\partial z^2} + k^2 \right) A_3(z) = j \omega \epsilon_0 \mu_0 E_3(z) \]

\[ \cdots \]
For finite \( A_3(3) \), singularity associated with \( E_3(3) \) at \( z = 0 \) must be accommodated by

\[
\frac{\partial^2}{\partial z^2} A_3(3).
\]

\[
\begin{align*}
\frac{1}{2} \int \omega e^{i\mu} & E_3(3) d z = -j \omega e^{i\mu} \left. \frac{d^2 A_3}{d z^2} \right|_3^3 + j \omega e^{i\mu} \\
&\quad - j \omega e^{i\mu} \\
&\quad - j \omega e^{i\mu}
\end{align*}
\]

so the first derivative of \( A_3 \) must change by

\[-j \omega e^{i\mu} \text{ across the source, for a unit strength source.}\]
\[
\frac{d}{dz} \left( \xi_1 \cos k z + \xi_2 \sin k z \right) = -\xi_1 k \sin k z + \xi_2 k \cos k z, \quad z > 0
\]
\[
\frac{d}{dz} \left( \xi_3 \cos k z + \xi_4 \sin k z \right) = -\xi_3 k \sin k z + \xi_4 k \cos k z, \quad z < 0
\]
\[z = \pm \frac{\delta}{2} \Rightarrow 0, \quad k (\xi_2 - \xi_4) = -j \omega \varepsilon_0 \omega \xi_2 - \xi_4 = -j \omega \varepsilon_0 \omega \frac{\mu_0}{k} = -j \frac{\mu_0}{\mu_0} \]

By symmetry \( \xi_2 = -\xi_4 = -j \frac{\mu_0}{\mu_0} \)

and \( \xi_1 = \xi_3 = 0 \).
Why is this?

Symmetry forces \( K_3(3) \) to be even, then

\[
A_3(3) = \frac{1}{4\pi} \int \frac{K_3(3') e^{-jkR}}{R} \, a \, d\phi' \, d3'
\]

forces \( A_3 \) to be symmetric as well.
Functional form of $A_3$

$$A_3(z) = C \cos k z - j \frac{\omega \epsilon_0}{2k} \sin k |z|$$

$$\frac{d}{d z^2} A_3 = -j \omega \epsilon_0 \left. \frac{d A_3}{d z^2} \right|_{z=\pm \frac{\epsilon}{2}}$$

get $C$ from

$$I(3) \Big|_{z \pm \frac{\epsilon}{2}} = 0$$