How to Solve?

\[ (\nabla_{\text{r}} \cdot \nabla_{\text{r}'} + k^2) \int_{\text{rod}} \frac{-jkR}{4\pi R} \text{d}V' = j \omega \varepsilon_0 \varepsilon \bar{E} \]

Pocklington's Equation

\[ \frac{h/2}{4\pi R} \int_{\text{rod}} \frac{-jkR}{4\pi R} I(z') e^{j k |z - z'|} \text{d}z' = C \cos k z - j \frac{\mu_0}{2k} \sin k |z| \]

\[ -h/2 \]

\[ R^2 = a^2 + (z - z')^2 \]

* Current on z-axis, observation point on dipole, or v.v. Hallen's Equation
Solutions to Hallen's Integral Equation

Rewrite equation in the form

\[ \frac{2}{\mu} \int G(z, z') I(z') \, dz' = C \cos k_z - \frac{j \mu}{2 \eta} \sin k \theta \]

where

\[ G(z, z') = \frac{e^{-jkr}}{4\pi r} + \frac{e^{-jkr'}}{4\pi r'} \]

\[ r = \sqrt{a^2 + (z-z')^2}, \quad r' = \sqrt{a^2 + (z+z')^2} \]
Moment Method solution

\[ \int_{a}^{b} G(z, z') f(z') \, dz' = q(z) \quad \text{F. 14. 1st kind} \]

Expand \( f(z) \) as \( c_1 f_1(z) + c_2 f_2(z) + \cdots + c_N f_N(z) + \hat{f}(z) \)

\( f_n(z) \) are basis functions chosen heuristically

\[ \sum_{n=1}^{N} c_n \int G(z, z') f_n(z') \, dz' = q(z) \]

Forgiving of form of \( \hat{f}(z) \) because of the integration. \( \ldots \)
While the true rule $f(z)$ gives equality for every value of $z$, \( \hat{f}(z) \) cannot do so. But it can do so approximately.

This approach is a great simplification of classical orthogonal expansions, such as Fourier expansions.

It is also extremely efficacious!!

(and pretty accurate as well)
Point Matching Solution - the present form of
\[ M(\beta) - \text{enforces equality at fixed points}, \]
\[ z_m, \]
\[ \sum_{n=1}^{\infty} c_n \int_{a}^{b} G(z_m, z') f_n(z') \, dz' = g(z_m) \]
\[ \sum a_{mn} c_n = b_m, \quad c_n \text{ are unknowns} \]
\[ a_{mn} = \int_{a}^{b} G(z_m, z') f_n(z') \, dz', \quad b_m = g(z_m) \]

in this case.

\[ \text{match point index} \]
\[ \text{source point index} \]
(more general) **Testing function Approach**

Also define a set of functions \( \{W_i\} = W_1, W_2, \ldots, W_N \). \( W_i \) are called testing functions or weighting functions.

\[
\sum_{n=1}^{N} c_n \int_{a}^{b} \int_{a}^{b} G(z, z') f(z') W_n(z') dz' dz = \int_{a}^{b} g(z) W_n(z) dz
\]

\[
\sum_{n=1}^{N} c_n a_{mn} = b_m \quad \text{or} \quad [a_{mn}] c_n = b_m
\]
Galerkin's Method

\[ W_m = f_m \]

Point-Matching Method

\[ W_m(z) = \delta(z - z_m) \]

But other choices are possible.
MOM

- based on mathematics of linear vector spaces and includes the theory of errors

- still an active area of research

- generally validated by experiments

- being displaced by finite element methods
  - sparse matrices
  - efficient computation
Example: Application to Hagen's Eq. for cylinder.

Consider

\[ Z_n = \frac{N-1}{N} \frac{L}{2} \]

\[ \Delta Z_n = Z_{n+1} - Z_n \text{ is } n^{th} \text{ sub-interval} \]

Use sub-sectional basis \( p_n(z) = \begin{cases} 1 & z \in \Delta Z_n \\ 0 & z \notin \Delta Z_n \end{cases} \) these are just pulses!!
... 

\[ I(\gamma) = \sum_{n=1}^{N} c_n \phi_n(\gamma) \quad \text{not a complete set, but an orthogonal set!} \]

We will attempt to solve for current on the dipole in terms of the stepped pulse approximation. This corresponds physically to an approximation by a series of ideal dipoles.

For example

\[ I(\gamma) \]

height is \( \gamma \)
How to form the equations

choose $Z_m$ at midpoint of $DZ_m$.

$$Z_m = \left( \frac{Z_{m-1}}{2N} \right)^{1/2}$$

divide length into $2N$ intervals and then take only the odd points, beginning with 1.

$$\sum_{n=1}^{\frac{1}{2}} c_n \int_{Z_n}^{Z_{n+1}} G(z_m, z') p_n(z') dz' = C \cos k \frac{Z_m}{2N} - \frac{i}{2N} \sin k |Z_m|$$

$$A_3 (\frac{Z_m}{2N})$$

$$a_{mn} = \int_{Z_n}^{Z_{m+1}} G(z_m, z') dz'$$
\[ P_n(z) \]

\[ I_n(z) \quad \text{for } z \in \mathbb{Z}_m \]

\[ |\frac{e^{-jkr}}{4\pi r}| \]

Sum of all such effects, \( P_n(z) \) on \( A_z(z_m) \)

must equal total \( A_z(3m) \)
\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN}
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  \vdots \\
  c_N
\end{bmatrix}
= 
\begin{bmatrix}
  C \cos \frac{k \pi}{2} - \frac{j \mu}{2 \eta} \sin \frac{k \pi}{2} \\
  \vdots \\
  C
\end{bmatrix}
\]

B.C. is \( c_N = 0 \)

So this can be re-arranged and simplified further, as \( \cdots \)
\[
\begin{bmatrix}
 a_{11} & a_{12} & \cdots & a_{1N-1} & d_1 \\
 a_{21} & a_{22} & \cdots & a_{2N-1} & d_2 \\
 \vdots & & \ddots & \vdots & \vdots \\
 a_{N1} & a_{N2} & \cdots & a_{NN-1} & d_N \\
\end{bmatrix}
\begin{bmatrix}
 c_1 \\
 c_2 \\
 \vdots \\
 c_N \\
\end{bmatrix} = 
\begin{bmatrix}
 b_1 \\
 b_2 \\
 \vdots \\
 b_N \\
\end{bmatrix}
\]

\[d_m = -\cos k z_m\]

\[b_m = -j/2 \mu \sin k z_m\]

find \( c_1, \ldots, c_{N-1}, \) \( C \) by standard methods.