Solutions For Homework #3

Problem 1:[10 pts]

(a) The diagram in Figure 1 shows the two-dimensional function $f(x,y) = \text{rect}(x-2, y-3) + 3\text{rect}(x-2, y-1)$ and the two projections $P_0\ f(x,y)$ and $P_{90}\ f(x,y)$. We can evaluate the projection $P_0\ f(x,y)$ mathematically

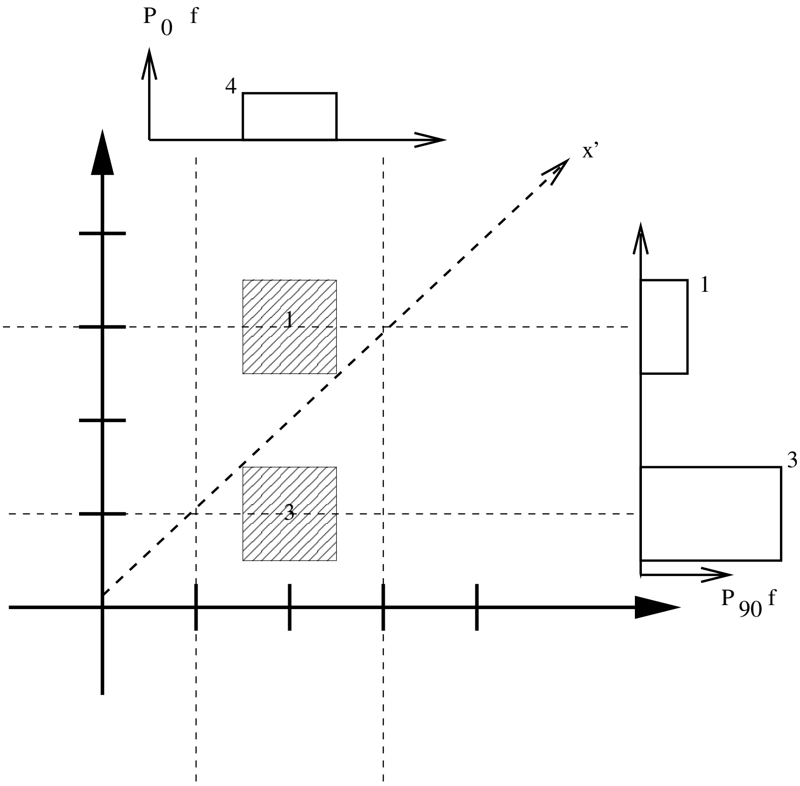

Figure 1:

as follows

$$
\begin{aligned}
P_0(R) &= \int_{-\infty}^{\infty} f(R,y)\,dy \qquad (1)\\
&= \int_{-\infty}^{\infty} \text{rect}(R-2,y-3) + 3\text{rect}(R-2,y-1)\\
&= 4\text{rect}(R-2)
\end{aligned}
$$

where $P_0(R)$ is the projection of the two dimensional function $f(x,y)$ along $x = R$.

(b) Figure 1 shows the two dimensional function $f(x,y) = \text{rect}(x-2,y-3) + 3\text{rect}(x-2,y-1)$ and the projection along a line parallel to the y-axis, $P_{90}\ f(x,y)$. Analogous to part (a), we compute the projection mathematically as follows

$$
\begin{aligned}
P_{90}(R) &= \int_{-\infty}^{\infty} f(x,R)\,dx \qquad (2)\\
&= \int_{-\infty}^{\infty} \text{rect}(x-2,R-3) + 3\text{rect}(x-2,R-1)\\
&= \text{rect}(R-3) + 3\text{rect}(R-1)
\end{aligned}
$$

where $P_{90}(R)$ is the projection of the two dimensional function $f(x,y)$ along $y = R$.

(c) In Figure 1, we see a line oriented 45 degrees from the x-axis, counterclockwise, labeled x'. This in the projection direction from which the one dimensional function $P_{45}\ f(x,y)$ will be computed. The most direct way to compute this diagonal projection is to rotate the given $x-y$ coordinate system, counterclockwise, by 45 degrees. We find the corresponding location (x',y') of a point (x,y) in the original coordinate system as follows

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos 45^o \sin 45^o \\ -\sin 45^o \cos 45^o \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (3)
$$

Using the formula above, we can find the locations of the centers of the two squares, $(2,3)$ and $(2,1)$ respectively, shown in Figure 2 in the new $x'-y'$

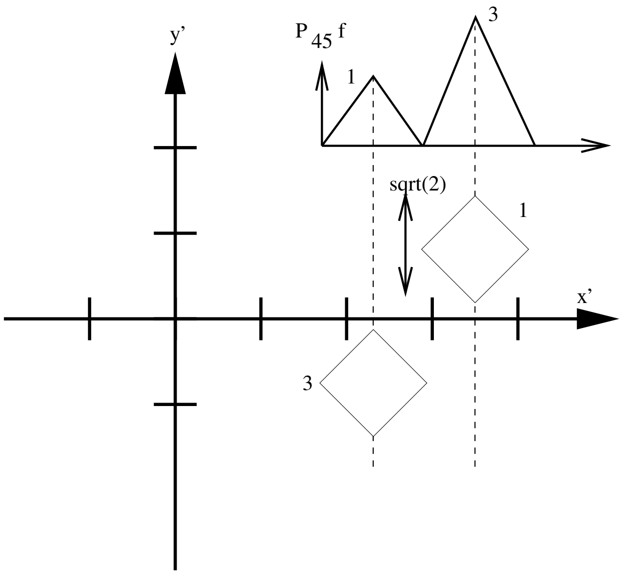

Figure 2:

coordinate system. We find that

$$(x,y) = (2,3) \quad \rightarrow \quad (x',y') = (\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \tag{4}$$
$$(x,y) = (2,1) \quad \rightarrow \quad (x',y') = (\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

By repeatedly applying the formula Equation 3 above to the (x, y) coordinates of the vertices of the two squares, we can deduce the shape of the objects in he $x' - y'$ coordinate system. This is shown in Figure 2. We know that the two "diamonds" are centered at $(\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ and $(\frac{3}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ respectively. Hence, the two projected functions will resembles triangles centered on $R = \frac{3}{\sqrt{2}}$ and $R = \frac{5}{\sqrt{2}}$ respectively, where $R = x'$, see Figure 2.

Defining a triangle function as follows

$$\Lambda(x) = \left\{ \begin{array}{cc} 1 - |x| & |x| < 1 \\ 0 & \text{else} \end{array} \right\} \tag{5}$$

we see that

$$P_{45} f(x,y) = 3\sqrt{2}\Lambda(\sqrt{2}(R - \frac{3}{\sqrt{2}})) + \sqrt{2}\Lambda(\sqrt{2}(R - \frac{5}{\sqrt{2}})) \tag{6}$$

Problem 2:[10 pts]

Figure 3 shows the result of backprojecting the 1-dimensional functions $P_0\ f(x,y)$ and $P_{90}\ f(x,y)$. The result of these two backprojection operations is the set of 4 objects that might comprise the desired function $f(x,y)$. In Figure 3, these 4 objects are denoted A,B,C and D respectively. We still need to apply backprojection to the remaining profile $P_{45}\ f(x,y)$ which will determine which of these four points are consistent with all three projection profiles.

As in Problem 1, we will map the (x,y) coordinates of these 4 candidate objects to a rotated coordinate system (x',y'). Subsequently, we will see that the projection profile $P_{45}\ f(x,y)$ can be obtained by integrating along the x'-direction. The four objects, A,B,C and D in the rotated $x'-y'$ coordinate system are shown in Figure 4 The profile $P_{45}\ f(x,y)$ is shown along with the 4 points. By inspection, we see that only points B,C and D can yield the 3 projection profiles $P_0\ f(x,y)$, $P_{90}\ f(x,y)$ and $P_{45}\ f(x,y)$.

Now that we have determined the number and locations of the three objects, we need to determine their intensities. Let S_i be the intensities of the three points B,C and D. Then,

$$\begin{aligned} S_C &= 0 \quad \text{from } P_{45}\ f(x,y) \\ S_A + S_D &= 1.5 \quad \text{from } P_{45}\ f(x,y) \\ S_B &= 0.5 \quad \text{from } P_{45}\ f(x,y) \\ S_C + S_D &= 0.5 \quad \text{from } P_{90}\ f(x,y) \end{aligned} \tag{7}$$

Solving the above equations, we find that $S_A = 1$, $S_B = 0.5$, and $S_D = 0.5$.

Problem 3:[10 pts]

We apply the three-step rule:

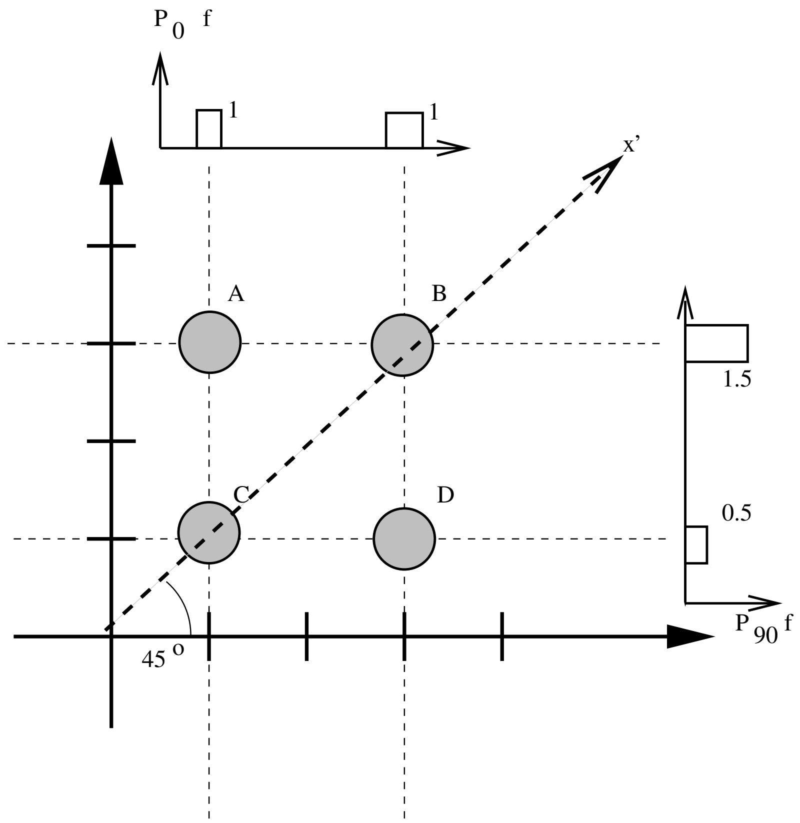

Figure 3:

(i) $\delta(f(x)) \rightarrow \frac{1}{\tau}\text{rect}\left(\frac{f(x)}{\tau}\right)$

(ii) perform operation

(iii) evaluate as $\tau \rightarrow 0$

$$\delta(f(x)) = \lim_{\tau \rightarrow 0} \frac{1}{\tau}\text{rect}\left(\frac{f(x)}{\tau}\right) \tag{8}$$

where

$$\text{rect}\left(\frac{f(x)}{\tau}\right) = \{ \begin{array}{cc} 1 & |x| < \frac{\tau}{2} \\ 0 & \text{else} \end{array} \tag{9}$$

Assuming τ is small, then rect $\left(\frac{f(x)}{\tau}\right)$ is non-zero only when the function $f(x)$ lies in a band of width τ about the x-axis. **We refer the reader to pages 3 and 4**

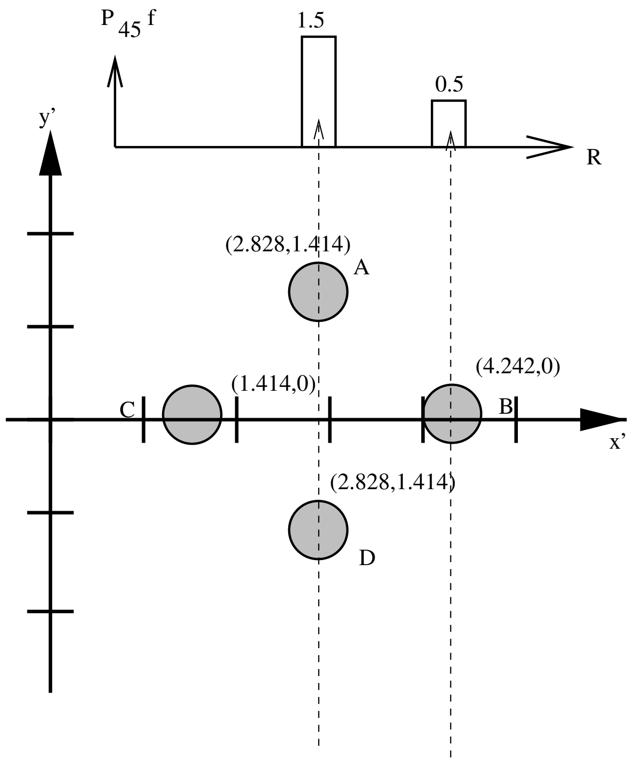

Figure 4:

of Handout 12 for a graphical explanation. Suppose, in particular, that in this function crosses the x-axis at x_0. In other words, let x_0 be a root of the function $f(x)$.

$$f(x_0) = 0 \tag{10}$$

We can expand this function as a Taylor series about this root. Furthermore, we keep only the first order term in the expansion.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 \\ &\approx f(x_0) + f'(x_0)(x - x_0) \end{aligned} \tag{11}$$

Subsituting the above expansion into the rect

$$\frac{1}{\tau}\text{rect}\left(\frac{f(x)}{\tau}\right) \approx \frac{1}{\tau}\text{rect}\left(\frac{f(x_0)+f'(x_0)(x-x_0)}{\tau}\right) \tag{12}$$

and making a change of variables $\lambda = \frac{\tau}{f'(x_0)}$, we have

$$\frac{1}{\lambda f'(x_0)}\text{rect}\left(\frac{(x-x_0)}{\lambda}\right) \tag{13}$$

Now, note that as $\tau \to 0$, $\lambda \to 0$. So,

$$\frac{1}{\lambda f'(x_0)}\text{rect}\left(\frac{(x-x_0)}{\lambda}\right) \to \frac{1}{f'(x_0)}\delta(x-x_0) \text{ as } \lambda \to 0 \tag{14}$$

Problem 4:[10 pts]

(a) The line integral in question can be expressed as follows

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(g(r))\,dx\,dy \tag{15}$$

where, $f(x,y) = cos(\pi x)$ and $g(r) = r - R$. To evaluate this, and subsequent integrals, we make use of our result from Problem 3 which, in 2 dimensions, can be stated as

$$\delta(f(x,y)) = \frac{1}{|\nabla f|}\delta(x-x_0, y-y_0) \tag{16}$$

where x_0 and y_0 denote a locus of points in the $x-y$ plane where the function $f(x,y)$ is zero. As the argument of the delta function is in polar coordinates, we should compute the polar coordinate gradient magnitude in order to find the strength of the impulse, $|\nabla(r-R)| = \sqrt{\left(\frac{\partial(r-R)}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial(r-R)}{\partial\theta}\right)^2} = 1$. Thus, the impulse $\delta(r-R)$ has unit strength.

$$\begin{aligned} I &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cos(\pi x)\delta(r-R)\,dx\,dy \qquad (17)\\ &= \int_{0}^{2\pi}\int_{0}^{\infty} \cos(\pi r\cos(\theta))\delta(r-R) r dr d\theta \\ &= \int_{0}^{\infty}\left[\int_{0}^{2\pi} \cos(\pi r\cos(\theta))\right] d\theta\delta(r-R) r dr \end{aligned}$$

where the second line in the above results from a change of Cartesian to Polar coordinates. Now, the θ integral, in square brackets above, is reminiscent of the definition of a Bessel function of first kind, order zero, defined as follows

$$\begin{aligned} J_0(x) &= \frac{1}{\pi}\int_0^{\pi} \cos(x\sin(\theta))d\theta \qquad (18) \\ &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos(x\cos(\theta'))d\theta', \quad \theta' = \theta - \frac{\pi}{2} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(x\cos(\theta''))d\theta'', \quad \theta'' = 2\theta' \end{aligned}$$

Thus, subsituting the Bessel function above in the line integral, we have

$$\begin{aligned} I &= \int_0^{\infty} 2\pi J_0(\pi r)\delta(r-R)r\,dr \qquad (19) \\ &= 2\pi R J_0(\pi R) \end{aligned}$$

by using the sifting property of delta functions.

(b) In this problem, the line integral is as follows

$$I = \int_0^{2\pi}\int_0^{\infty} \cos(\pi r\cos(\theta))\delta(r^2 - R^2)r\,dr\,d\theta \qquad (20)$$

We first need to compute the strength of the impulse, using our result from problem 3. The gradient magnitude (in polar coordinates) of the argument of the delta-function in this part is

$$|\nabla(r^2 - R^2)| = 2r \qquad (21)$$

which gives us the strength of the impulse as $\frac{1}{2r}$. Furthermore, the function $r^2 - R^2 = 0$ only when $r = R$. Consequently, the delta function can be expressed as

$$\delta(r^2 - R^2) = \frac{1}{2R}\delta(r - R) \qquad (22)$$

Now, performing the θ integral first, plugging in the expression for the delta function above and recalling the definition of the Bessel function we used in part (a),

$$\begin{aligned} I &= \int_0^{\infty} 2\pi J_0(\pi r)\frac{1}{2R}\delta(r-R)r\,dr \qquad (23) \\ &= \pi J_0(\pi R) \end{aligned}$$

by the sifting property of delta functions.

(c) In this problem, the line integral is as follows

$$I = \int_0^{2\pi} \int_0^{\infty} \cos(\pi r \cos(\theta))\delta(tan^{-1}(r-R))rdrd\theta \tag{24}$$

The strength of the impulse function is proportional to the gradient magnitude of its argument

$$|\nabla(\tan^{-1}(r-R))| = \frac{1}{1+(r-R)^2} \tag{25}$$

This gradient magnitude, a function of r, needs to be evaluated at values of r such that $\tan^{-1}(r-R) = 0$. Of course, the arctangent is zero only when $r = R$, which leads us to find that the strength of the delta function is unity. Thus, the line integral expression now becomes

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^{\infty} \cos(\pi r \cos(\theta))\delta(r-R)rdrd\theta \\ &= 2\pi R J_0(\pi R) \end{aligned} \tag{26}$$

(d) The strength per unit length of the impulse function $\delta((r-R)^2)$ is computed from the magnitude of the gradient of the argument of the delta function,

$$|\nabla(r-R)^2| = 2(r-R) \tag{27}$$

This gradient magnitude is evaluated at the locus of points where $(r-R)^2 = 0$. Clearly, this locus is a circle of radius R. However, we find that, on this circle, the gradient magnitude is zero yielding an infinite strength per unit length. To understand this, we approximate the impulse function by a rect

$$\delta((r-R)^2) \approx \frac{1}{\tau}\text{rect}\left(\frac{(r-R)^2}{\tau}\right) \tag{28}$$

Note that for a given value of τ,

$$\text{rect}\left(\frac{(r-R)^2}{\tau}\right) = \left\{ \begin{array}{cc} 1 & (r-R)^2 < \tau/2 \\ 0 & \text{else} \end{array} \right\} \tag{29}$$

This means that the width of the rect function is $2\sqrt{\frac{\tau}{2}}$. Consequently, the area of the rect function is $\frac{1}{\tau} \times 2\sqrt{\frac{\tau}{2}} = \sqrt{\frac{2}{\tau}}$. The area, therefore, is not

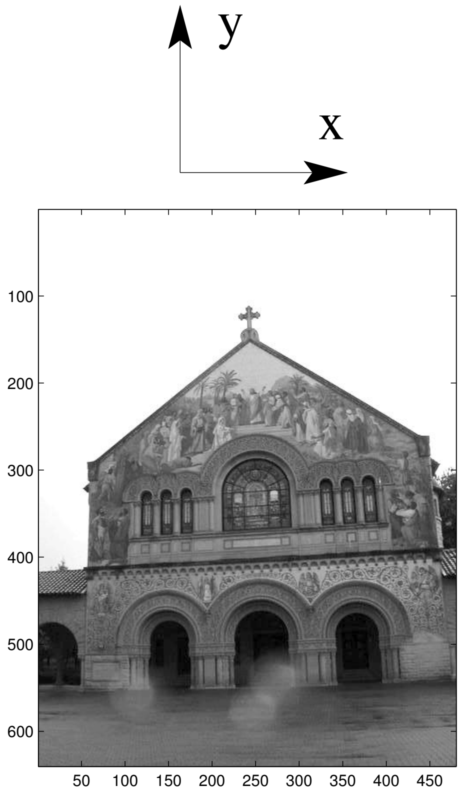

Figure 5:

independent of τ and increases to infinity as τ becomes infinitesimal. As the area the rect function is proportional to the strength of the delta, we see then why $\delta((r - R)^2)$ has infinite strength.

Problem 5:[10 pts]

(a) The image *hw3p5image* is shown in Figure 5. Figure 5 shows a grayscale image of Memorial Church. In addition, a $y-x$ coordinate system is shown. We need to establish a mapping between row/column - (i, j) - pixel indices and (x, y) coordinates, under the assumption that the pixel spacings in both dimensions are unity.

$$x = j - 241 \qquad (30)$$
$$y = 321 - i$$

Thus, evaluating the line integral of the product of the image with the impulse function $\delta(x)$ is simply the sum of all pixel values in column 241.

$$I = \sum_{i=1}^{640} \text{image}(i, 241) = 87164 \qquad (31)$$

(b) Evaluating the line integral of the product of the image with the impulse function $\delta(2y) = \frac{1}{2}\delta(y)$ is simply the sum of all pixel values in row 321, scaled by one half

$$I = \frac{1}{2}\sum_{j=1}^{480} \text{image}(321, j) = 28433 \qquad (32)$$

(c) In this part, we sum all pixel values located on the line $y - x = 0$ or $(321 - i) - (j - 241) = 562 - i - j = 0$.

$$I = \sum_{k} \text{image}(k, 562 - k) = 69628 \qquad (33)$$

We do not need to scale this sum, I, by the strength per unit length of the line impulse $\delta(y - x)$ because, in this case, the strength per unit length is unity. Recall from the class notes, Handout 11, page 12, that the strength of a unit length impulse situated on a line $y = mx + c$ is $\cos\alpha$, where α is the angle that the line makes with the horizontal x-axis. So, in this problem, the strength per unit length of that line impulse would be $\cos 45^o = \frac{1}{\sqrt{2}}$. Figure 6 shows a unit length rect function of width τ approximating the line impulse drawn on a grid with unit spacing. The grid represents the pixel points of the image which we wish to sample. We note that the minimum diagonal distance between two pixels is $\sqrt{2}$. Because of this, the strength per unit length of $\frac{1}{\sqrt{2}}$ of our line impulse $\delta(y-x)$ is multiplied by a distance of $\sqrt{2}$ to give a strength per pixel of 1.

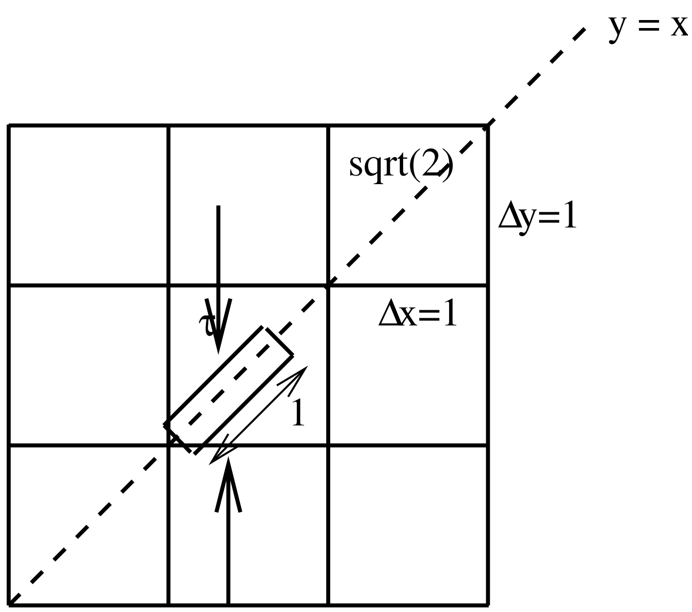

Figure 6:

(d) The line integrate we wish to evaluate is, in polar coordinates,

$$I = \int_0^{2\pi} \int_0^{\infty} f(r\cos\theta, r\sin\theta)\delta(r-100)\ r\ dr\ d\theta \qquad (34)$$

We approximate the above integral by a sum,

$$I \approx \sum_i \sum_{j=1}^{\infty} f(r_j\cos\theta_i, r_j\sin\theta_i)\delta(r_j-100)r_j\Delta r\Delta\theta \qquad (35)$$

where the i summation in the above is over the number of distinct angles that can be formed using the unit spacing $x-y$ grid of the digital image. Of course, $f(r_j\cos\theta_i, r_j\sin\theta_i)$ in the above refers to the ij-th pixel value of the image. We assume in the above that there exists a certain bin j for which $r_j = 100$. By virtue of the delta function above, we can then eliminate the inner, j, summation

$$I \approx \sum_{i=1}^{N} f(100\cos\theta_i, 100\sin\theta_i)100\Delta\theta \qquad (36)$$

At this point, we assume that the length of the element of arc, a distance 100 pixels from the origin, is unity, i.e. $100\Delta\theta = 1$. This allows us to find N, the number of distinct angles, in the above. We mention here that different assumptions regarding arc length will yield different values of N, thereby affecting the final sum. In our case, we find that $I = 75887$

Note: Using a more direct method of finding all pixels in the image that have distance 100 from the origin will yield a sum of 77723. This method is acceptable as well.

Problem 6:[10 pts]

(a) To calculate the strength per unit area of the impulse $\delta(r - \sqrt{x^2 + y^2 + z^2})$, we apply the three-step rule

(i) $\delta(f(r)) \rightarrow \frac{1}{\tau}\text{rect}\left(\frac{f(r)}{\tau}\right)$

(ii) perform operation

(iii) evaluate as $\tau \rightarrow 0$

Now, in three dimensions, $\frac{1}{\tau}\text{rect}\left(\frac{r-\sqrt{x^2+y^2+z^2}}{\tau}\right)$ can be visualized as a shell of radius $R = \sqrt{x^2 + y^2 + z^2}$ and thickness τ. An intensity of $\frac{1}{\tau}$ is uniformily distributed at all points on this shell. The volume of this shell, V, can be computed as follows

$$V = \frac{4}{3}\pi(R + \frac{\tau}{2})^3 - \frac{4}{3}\pi(R - \frac{\tau}{2})^3 = \frac{4}{3}\pi(\frac{\tau^3}{4} + 3R^2\tau) \tag{37}$$

Finally, the total mass of this shell is $V \times \frac{1}{\tau} = \frac{4}{3}\pi(\frac{\tau^2}{4} + 3R^2)$. Applying the third rule, we take the limit as τ goes to zero and find that the mass of a shell of thickness τ is exactly $4\pi R^2$. But, this mass is distributed across a surface area of $4\pi R^2$ as well. This means that the strength per unit area of the impulse $\delta(r - R)$, in three dimensions, is unity as it was in the 2D case.

The total mass (strength) integrated over all space is

$$\begin{aligned} M &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(r - R)dx\, dy\, dz \\ &= \int_{0}^{\infty}\int_{0}^{2\pi}\int_{0}^{\pi} \delta(r - R)r^2 \sin\theta d\theta d\phi dr \\ &= 4\pi R^2 \end{aligned} \tag{38}$$

(b) The isotropic intensity pattern we are given is

$$f(r) = \frac{100 \text{ Watts}}{4\pi r^2}, \quad r = \sqrt{x^2 + y^2 + z^2} \tag{39}$$

We find the strength per unit area of the delta function $\delta(r - R)$ by computing the magnitude of the gradient at $r = R$

$$|\nabla (r - R)|_{r=R} = 1 \tag{40}$$

Thus, the strength per unit area of the delta function, $S(r)$ is equal to unity.

Recording the total amount of power emitted by this light source at a certain distance R is equivalent to summing up the power densities on the surface of a shell of radius R, weighted by the strength per unit area $S(r)$ of the delta function. Mathematically,

$$\begin{aligned} I &= \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi} f(r) S(r) \delta(r - R) r^2 \sin\theta d\theta d\phi dr \qquad (41) \\ &= \frac{100}{4\pi R^2} \int_0^{2\pi} d\theta \int_0^{\pi} R^2 \sin\phi \; d\phi \\ &= 100 \end{aligned}$$

Thus, we see that $I = 100$ Watts, independent of the distance R where the observation is made. This makes physical sense because, by conservation of energy, the 100 Watts radiated by the light source should be maintained in a spherical shell of any radius R. Of course, the power *density* decreases as $\frac{1}{R^2}$ with increasing distance from the location of the light source, as the surface area of that shell increases as $4\pi R^2$.

MATLAB code for Problem 5

```
% load in image file, 480 rows by 640 cols

numrows = 480; numcols = 640;

fid = fopen('hw3p5image','rb');
data = fread(fid,inf,'uint8');
```

```
fclose(fid);

im = reshape(data,[numrows numcols]); im = im';

% Note: x is in column direction, y is in row direction
Oy = 321; Ox = 241;

% create grid
[x,y] = meshgrid((1:numrows)-Ox, (numcols:-1:1)-Oy);

% part(a): delta(x)
%-----------------%

Ia = sum(im(:,Ox));

% part(b): delta(2y)
%-----------------%

Ib = sum(im(Oy,:))/2;


% part(c): delta(x-y)
%-----------------%
mask = (x==y); ind = find(mask == 1);
Ic = sum(im(ind));

% part(d): delta(r - 100)
%-----------------%

theta = 0:1/100:2*pi;
xx = round(100*cos(theta)); yy = round(100*sin(theta));
Id = sum(im(sub2ind(size(im),Oy-yy,xx+Ox)))
```