# Solutions For Homework #5

## Problem 1:[15 pts]

The function $f(x,y) = e^{-\pi\left(\frac{x^2}{a^2}+y^2\right)}$ has a Fourier Transform as follows

$$f(x,y) \supset |a|e^{-\pi(a^2u^2+v^2)} \tag{1}$$

which is obtained by applying the Similarity Theorem to the Fourier Transform of a two-dimensional Gaussian. Now, we seek the function $h(x,y)$ such that the function

$$g(x,y) = (f**g)(x,y) \tag{2}$$

depends only on $\sqrt{x^2+y^2}$; that is, $g$ should be circularly symmetric. By the Convolution Theorem, this means that the Fourier Transform of $g(x,y)$

$$G(u,v) = H(u,v)F(u,v) \tag{3}$$

is also circularly symmetric, i.e depends only on the argument $\sqrt{u^2+v^2}$.

Suppose $|a| < 1$.

Consider $H(u,v) = e^{+\pi a^2 u^2} e^{-\pi u^2} \cdot 1 = e^{-\pi u^2(1-a^2)}$. Then,

$$G(u,v) = e^{-\pi(u^2+v^2)} \quad \supset \quad g(x,y) = e^{-\pi(x^2+y^2)} \tag{4}$$

are circularly symmetric functions, as desired, and the convolution kernel used is

$$H(u,v) \quad \supset \quad h(x,y) = \frac{1}{\sqrt{1-a^2}} e^{-\pi\frac{x^2}{1-a^2}} \delta(y) \tag{5}$$

Suppose $|a| > 1$.

Consider $H(u,v) = e^{+\pi v^2} e^{-\pi a^2 v^2} \cdot 1 = e^{-\pi v^2(a^2-1)}$. Then,

$$G(u,v) = e^{-\pi(a^2u^2+a^2v^2)} \quad \supset \quad g(x,y) = \frac{1}{a^2} e^{-\pi(\frac{x^2}{a^2}+\frac{y^2}{a^2})} \tag{6}$$

are circularly symmetric functions, as desired, and the convolution kernel $h(x,y)$ can be found by taking the inverse Fourier Transform of its transfer function:

$$H(u,v) \quad \supset \quad h(x,y) = \frac{1}{\sqrt{a^2-1}} e^{-\pi\frac{y^2}{a^2-1}} \delta(x) \tag{7}$$

**Problem 2:**[15 pts]

(a) We let the aperture illumination function be denoted $f(x) = \text{rect}(x)$. From the Fraunhofer approximation, we know that the power pattern of the antenna in the far-field is proportional to the magnitude-squared of the Fourier Transform of the aperture illumination function. Mathematically, if $P(\theta)$ is the power pattern in the far-field,

$$P(\theta) = \left| \int_{-\infty}^{\infty} f(x) e^{-i2\pi x \frac{\sin\theta}{\lambda}} \, dx \right|^2 \tag{8}$$

Note that the "frequency" variable in the above is given by $\frac{\sin\theta}{\lambda}$. Here, $\lambda = 0.1$ meters. As we know that

$$\text{rect}(x) \;\supset\; \text{sinc}(s) \tag{9}$$

then, the far-field power pattern of the antenna is given by $P(\theta) = \text{sinc}^2\left(\frac{\sin\theta}{\lambda}\right)$. Figure 1 shows this power pattern on a dB-scale, i.e. the vertical axis in Figure 1 is in units of $10\log_{10} P(\theta)$. The horizontal axis is angle $\theta$ measured in degrees. Remember that the argument of the sinc-squared function that forms the power pattern above is $\frac{\sin\theta}{\lambda}$.

(b) We measure the sidelobe ratio directly from our plot in Figure 1. As is shown, this ratio is about $13.6$ dB, or the central peak is about 22.9 times greater than the highest sidelobe.

(c) Figure 2 shows the aperture illumination function $\text{rect}(x)$ along with the tapering function, a raised cosine. We model this tapering function $h(x)$ as follows

$$h(x) = \frac{1}{2}(1 + \cos(2\pi x)) \tag{10}$$

As $\text{rect}(x)$ is nonzero only for $0.5 < x < 0.5$, then we see that at the edges of the aperture, $|x| = 0.5$, $h(x) = 0$, as desired. Figure 3 shows a comparison between the far-field patterns that arise from the unweighted (solid line) and tapered (dashed line) illumination functions. **We observe that the peak sidelobe ratio for the tapered power pattern is about 32.25**

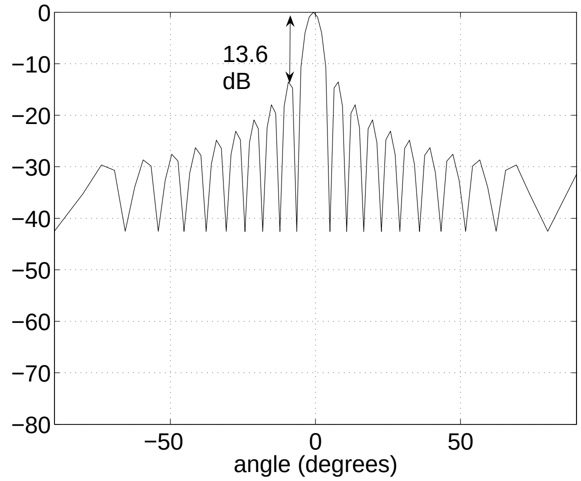


Figure 1:

**dB, or the central peak is about 1679 times greater than the highest sidelobe**. Observe also the broadening of the central peak. This is because of the effect of the taper placed on the aperture illumination function.

(d) By the convolution theorem, the tapered aperture has a Fourier Transform given by

$$\text{rect}(x)\frac{(1+\cos(2\pi x))}{2} \supset \text{sinc}(s) * \left(\frac{1}{2}\delta(s) + \frac{1}{4}\delta(s+1) + \frac{1}{4}\delta(s-1)\right) \tag{11}$$

The result of the convolution above is basically a weighted sum of three sincs, shifted with respect to one another. This is shown in Figure 4. Figure 4 shows 3 sinc functions, the largest of which is centered at 0 frequency. Frequency, denoted $s$ here, is in some, unspecified, units. It could be, for example, in units of radians per meter, as in part (b) above, where $s = \frac{sin\theta}{\lambda}$. Nonetheless, the three sinc functions shown in Figure 4 are shifted relative to another another. The magnitude-squared of the sum of these three sinc functions yields the power pattern with reduced sidelobes we saw in part (c). Essentially, the troughs of the shifted sincs' sidelobes (thick lines) coincide with the location of the sidelobe peaks of the largest, centrally-oriented, sinc function (dashed line). This relative orientation causes a cancellation of the sidelobes.

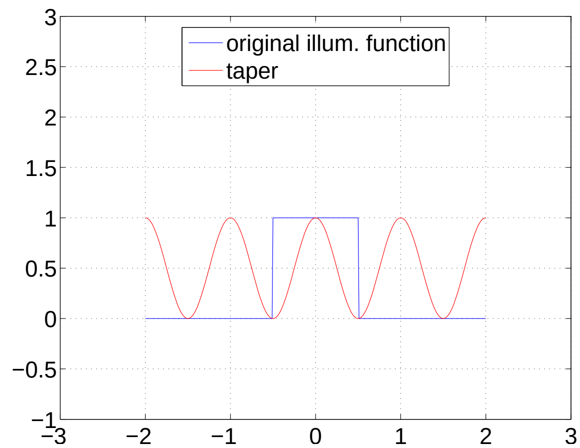


Figure 2:

## Problem 3:[15 pts]

(a) This problem is a two-dimensional version of problem 2. In this case, the wavelength of radiation is $\lambda = 0.25$ meters. The circular aperture is uniformily illuminated. Analogous to Problem 2, we know that the far-field power pattern is proportional to magnitude-squared of the Fourier transform of this aperture illumination function.

For 2D Fraunhofer problems, we need to define the geometry. This is shown in Figure 5, which shows the angles $\theta$ and $\gamma$. Under the Fraunhofer approximation and the defintion of $\theta$ and $\gamma$, then, the far-field power pattern is

$$P(\theta, \gamma) = \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-i2\pi\left(x\frac{\sin\gamma}{\lambda} + y\frac{\sin\theta}{\lambda}\right)} \, dx \, dy \right|^2 \qquad (12)$$

where $f(x, y)$ represents the aperture illumination function. From the equation above, the two "frequency" variables are $u = \frac{\sin\gamma}{\lambda}$ and $v = \frac{\sin\theta}{\lambda}$. For illumination functions that are circularly symmetric, as in our case where $f(x, y) = \text{rect}(r)$, the 2D problem effectively reduces to 1D. In particular, the Fourier transform above reduces to a Hankel transform

$$P(\phi) = \left| \int_{-\infty}^{\infty} f(r) e^{-i2\pi\left(r\frac{\sin\phi}{\lambda}\right)} \, r \, dr \right|^2 \qquad (13)$$

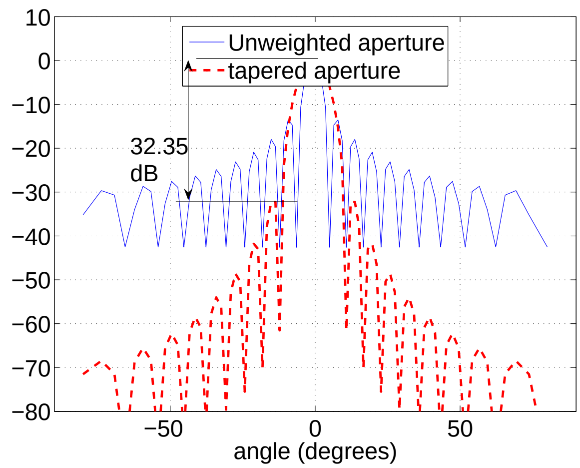


Figure 3:

where, now, the definition of the angle $\phi$ is shown in Figure 6. We know that the Hankel transform of rect$(r)$ is $jinc(q) = \frac{J_1(\pi q)}{2q}$ and so, the far-field power-pattern is

$$P(\phi) = \text{jinc}^2\left(\frac{\sin\phi}{\lambda}\right) \tag{14}$$

The far-field power pattern is shown in Figure 7. The vertical axis is in decibels (dB) or $10\log_{10} P(\theta)$. In two-dimensions, the power-pattern $P(\theta, \gamma)$ is plotted in Figure 8 in decibels (dB). Note that now, the horizontal axis denotes spatial frequency $u = \frac{\sin\gamma}{\lambda}$ while the vertical axis represents spatial frequency $v = \frac{\sin\theta}{\lambda}$ where the angles $\gamma$ and $\theta$ were defined previously.

(b) Analogous to Problem 2, we define the two-dimensional, circularly symmetric, tapering function $h(r)$ as follows

$$h(r) = \frac{1}{2}(1 + \cos(2\pi r)) \tag{15}$$

The weighted aperture illumination function is given by

$$g(r) = h(r)f(r) = \frac{1}{2}(1 + \cos(2\pi r))\text{rect}(r) \tag{16}$$

shown in Figure 9. The far-field power pattern is computed numerically by calculating the 2D Fourier Transform of the tapered illumination function,

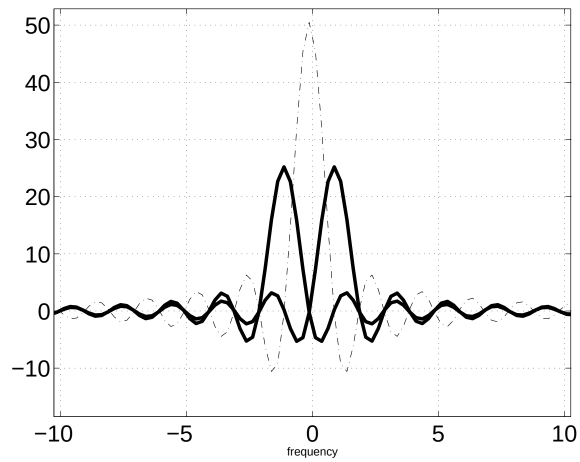


Figure 4:

Figure 9, and subsequently squaring its magnitude. This function is shown in Figure 10. Since the unweighted and tapered illumination functions generate power patterns that are circularly symmetric, we compare 1D slices through the origin of both 2D power patterns. This is to evaluate the peak-to-sidelobe ratios of the two power patterns. The cuts are shown in Figure 11 By inspection of the power profiles directly, we find that the **peak-to-sidelobe ratio of the power pattern corresponding to the unweighted aperture (solid line) is about 17.75 dB**. The **peak-to-sidelobe ratio of the power pattern corresponding to the tapered illumination function is found to be about 33.9 dB**.

## **Problem 4:**[15 pts]

(a) The function plotted in the figure can be described as a 2-D rect function convolved with four unit impulses. The four impulses themselves can be seen as a pair of impulses on the y-axis with amplitude +1 each, and a pair on the x-axis where the one at (2,0) has amplitude +1 and the one at (-2,0) has amplitude -1. Convolution in the spatial domain leads to multiplication in the frequency domain, so we can evaluate the transforms of the two components individually and then calculate their product.

The transform of rect($x, y$) is simply sinc($u, v$).

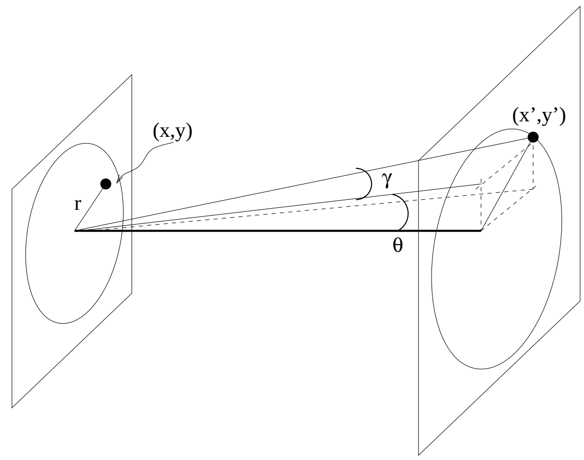


Figure 5:

The transform of the pair of impulses along the x-axis is $i * 2\ \sin 4\pi u$.

The transform of the pair along the y-axis is $2\ \cos 4\pi v$.

Thus the analytic transform is

$$(i * \sin 4\pi u + \cos 4\pi v)\ 4\ \mathrm{sinc}(u, v) \tag{17}$$

The numerical and analytic solutions are shown in Figures 12 and 13, respectively.

(b) Difference images are shown in Figures 14 and 15. Note that the solution is very close in the center of the image but degrades with distance from the center, where the magnitude of the numbers decreases.

## MATLAB code for Problem 4

```
%hw5prob4

clear all; close all;

q=sinc2(0,0)
q=sinc2(.01,.01)
q=sinc2(12.7, 23.1)
```

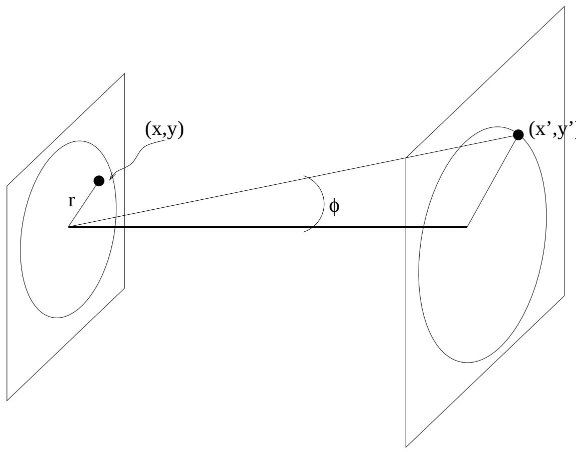


Figure 6:

```
a=zeros(401,401);

for k=-5:5
    for j=-5:5
        a(201-2*11+k, 201+j     ) =  1;
        a(201+k,      201+2*11+j) =  1;
        a(201+2*11+k, 201+j     ) =  1;
        a(201+k,      201-2*11+j) = -1;
    end
end

imagesc(a); colorbar;
title('grid');

s=fftshift(fft2(a/11^2));

figure;
imagesc(abs(s)); colorbar;
title('numerical');
print -depsc 'numerical';

maxreal=max(max(real(s)))
```

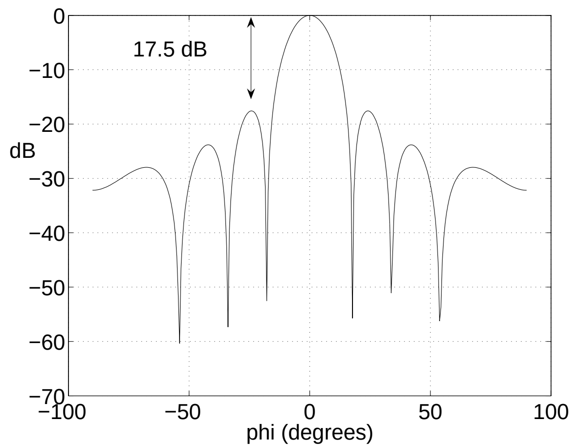


Figure 7:

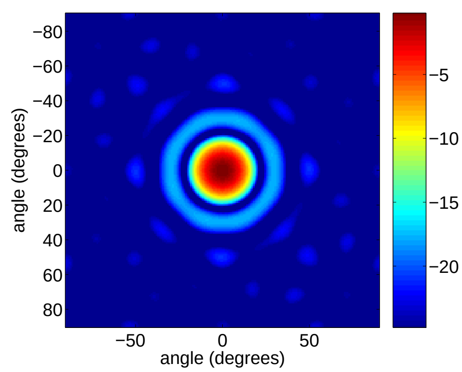


Figure 8:

```
maximag=max(max(imag(s)))

scale=401/11
for u=-200:200
    uu=u/scale;
    for v=-200:200
        vv=v/scale;
        analytic = (i*sin(4*pi()*uu)+cos(4*pi()*vv))*2*sinc2(uu,vv);
        s2(u+201,v+201)=analytic;
    end
end
```

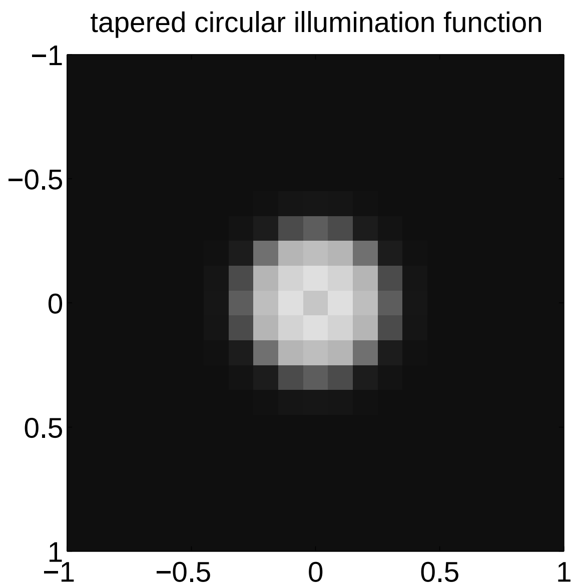


Figure 9:

```
s2t = abs(s2');
figure; imagesc(s2t); colorbar;
title('analytic');
print -depsc 'analytic';

diff = s2t - abs(s);

figure; imagesc(diff); colorbar;
title('difference');
print -depsc 'difference';

figure;
imagesc(abs(100*diff./s2t)); colorbar;
title('percent difference');
print -depsc 'percentdifference';
```

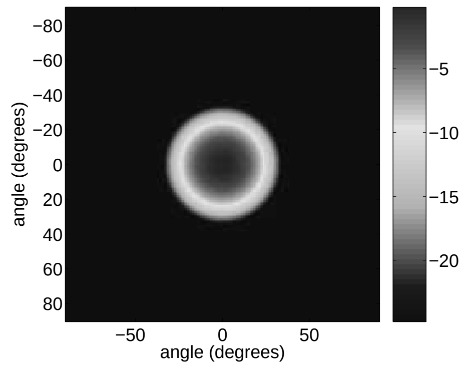


Figure 10:

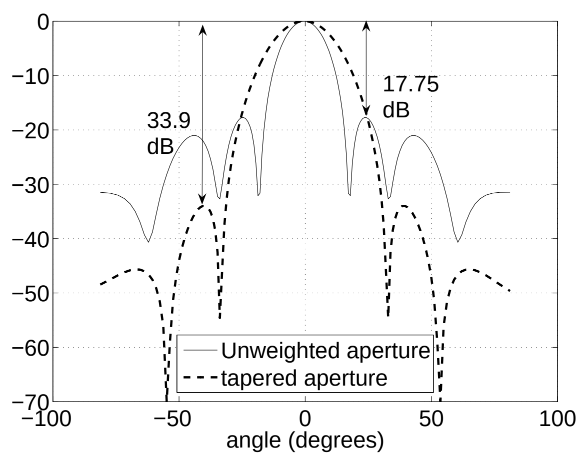


Figure 11:

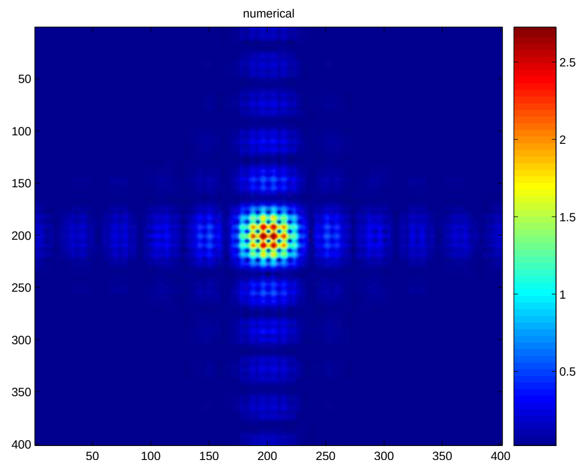


Figure 12:

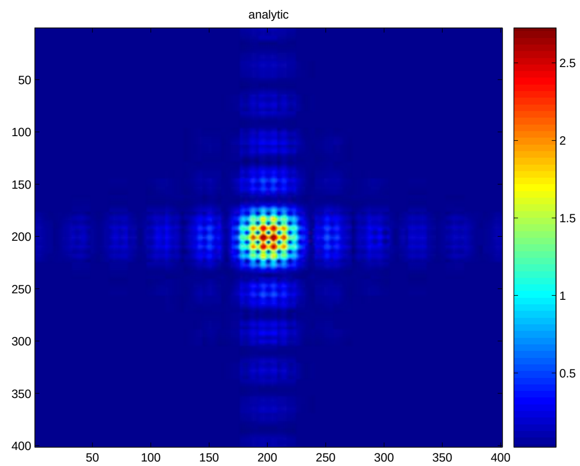


Figure 13:

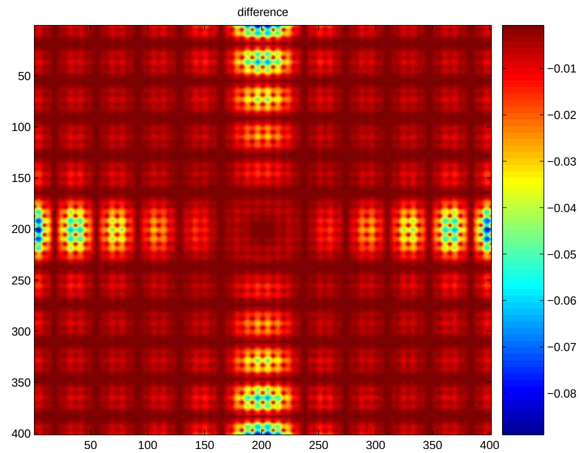


Figure 14:

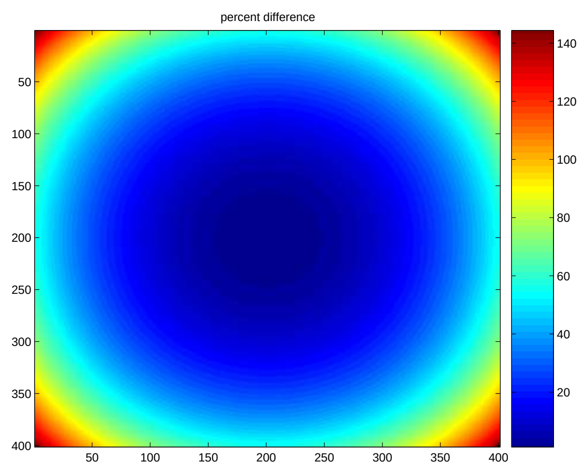


Figure 15: