Pose Tracking II

Gordon Wetzstein
Stanford University

EE 267 Virtual Reality

Lecture 12

stanford.edu/class/ee267/

WARNING

- this class will be dense!
- will learn how to use nonlinear optimization (Levenberg-Marquardt algorithm) for pose estimation
- why ???
    - more accurate than homography method
    - can dial in lens distortion estimation, and estimation of intrinsic parameters (beyond this lecture, see lecture notes)
    - LM is very common in 3D computer vision → camera-based tracking

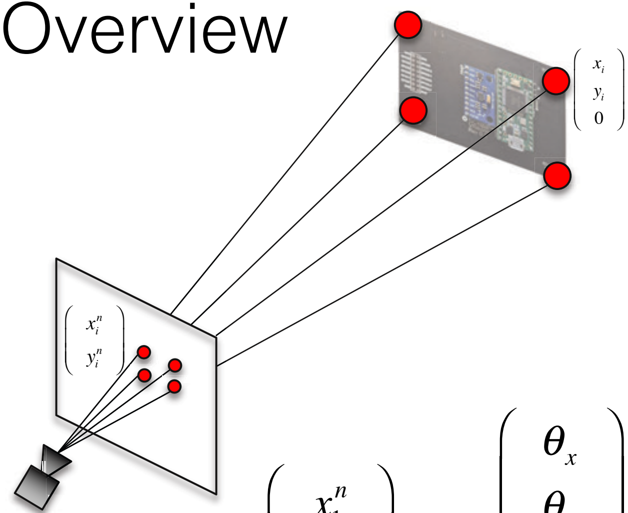
Pose Estimation - Overview

- goal: estimate pose via nonlinear least squares optimization

$$\underset{\{p\}}{\text{minimize}} \left\| b - f\left(g\left(p\right)\right) \right\|_2^2$$

↑
image formation

- minimize reprojection error
- pose *p* is 6-element vector with 3 Euler angles and translation of VRduino w.r.t. base station



$$b = \begin{pmatrix} x_1^n \\ y_1^n \\ \vdots \\ x_4^n \\ y_4^n \end{pmatrix} \qquad p = \begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \\ t_x \\ t_y \\ t_z \end{pmatrix}$$

Overview

- review: gradients, Jacobian matrix, chain rule, iterative optimization
- nonlinear optimization: Gauss-Newton, Levenberg-Marquardt
- pose estimation using LM
- pose estimation with VRduino using nonlinear optimization

Review

Review: Gradients

- gradient of a function that depends on multiple variables:

$$\frac{\partial}{\partial x} f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$$

$$f : \Re^n \to \Re$$

Review: The Jacobian Matrix

- gradient of a vector-valued function that depends on multiple variables:

$$\frac{\partial}{\partial x} f(x) = J_f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$$f : \Re^n \to \Re^m, \quad J_f \in \Re^{m \times n}$$

Review: The Chain Rule

- here's how you've probably been using it so far:

$$\frac{\partial}{\partial x} f\big(g(x)\big) = \frac{\partial f}{\partial g} \cdot \frac{\partial g}{\partial x}$$

- this rule applies when $f : \Re \rightarrow \Re$

$$g : \Re \rightarrow \Re$$

Review: The Chain Rule

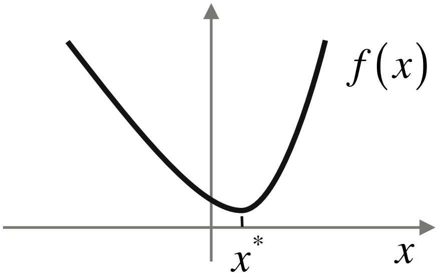
- here's how it is applied in general:

$$\frac{\partial}{\partial x} f\left(g(x)\right) = J_f \cdot J_g = \begin{pmatrix} \frac{\partial f_1}{\partial g_1} & \cdots & \frac{\partial f_1}{\partial g_o} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial g_1} & \cdots & \frac{\partial f_m}{\partial g_o} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_o}{\partial x_1} & \cdots & \frac{\partial g_o}{\partial x_n} \end{pmatrix}$$

$$f : \Re^o \rightarrow \Re^m, \quad g : \Re^n \rightarrow \Re^o, \quad J_f \in \Re^{m \times o}, \quad J_g \in \Re^{o \times n}$$

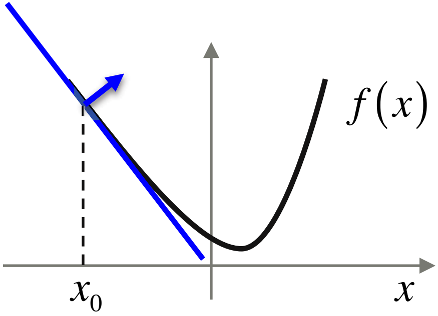
Review: Minimizing a Function

- goal: find point x^* that minimizes a nonlinear function $f(x)$



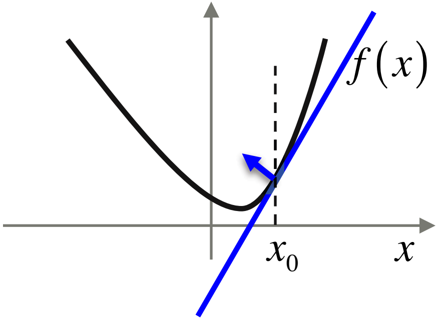
Review: What is a Gradient?

- gradient of *f* at some point x_0 is the slope at that point



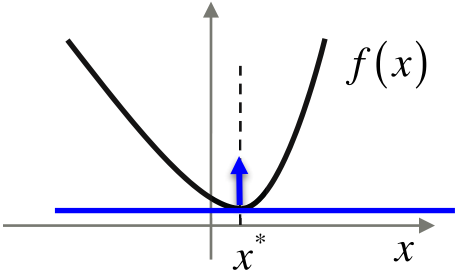
Review: What is a Gradient?

- gradient of f at some point x_0 is the slope at that point



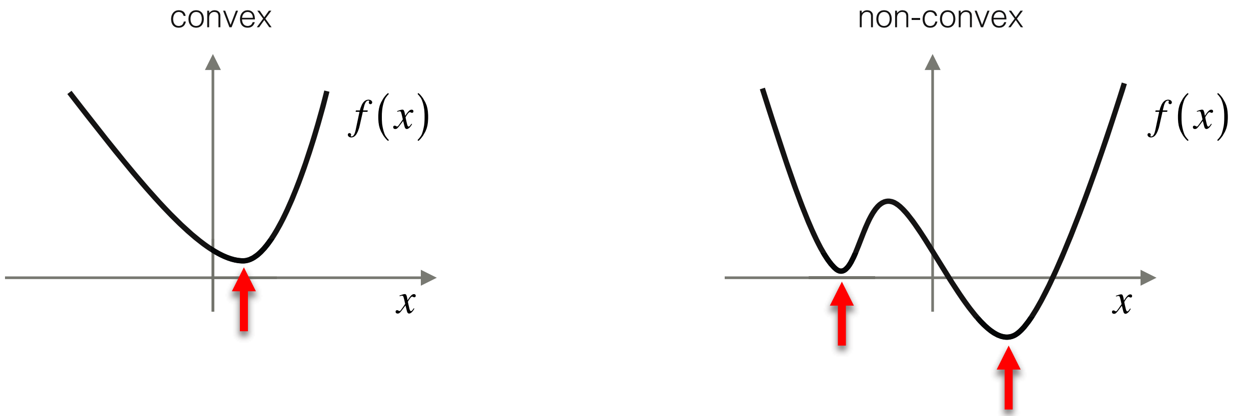
Review: What is a Gradient?

- extremum is where gradient is 0! (sometimes have to check 2nd derivative to see if it's a minimum and not a maximum or saddle point)



Review: Optimization

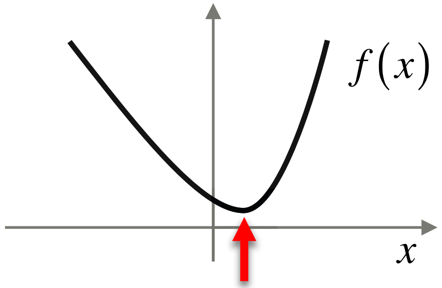
- extremum is where gradient is 0! (sometimes have to check 2nd derivative to see if it's a minimum and not a maximum or saddle point)



- convex optimization: there is only a single *global* minimum
- non-convex optimization: multiple *local* minima

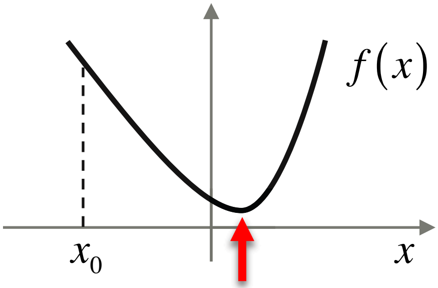
Review: Optimization

- how to find where gradient is 0?



Review: Optimization

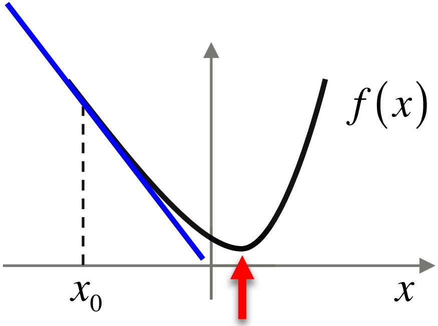
- how to find where gradient is 0?



1. start with some initial guess x_0, e.g. a random value

Review: Optimization

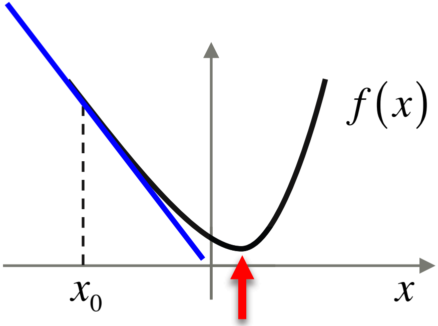
- how to find where gradient is 0?



1. start with some initial guess x_0, e.g. a random value
2. update guess by linearizing function and minimizing that

Review: Optimization

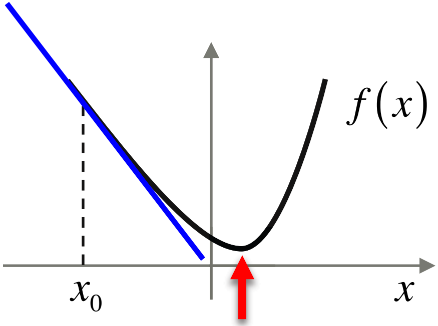
- how to linearize a function? → using Taylor expansion!



$$f(x_0 + \Delta x) \approx f(x_0) + J_f \Delta x + \varepsilon$$

Review: Optimization

- find minimum of linear function approximation



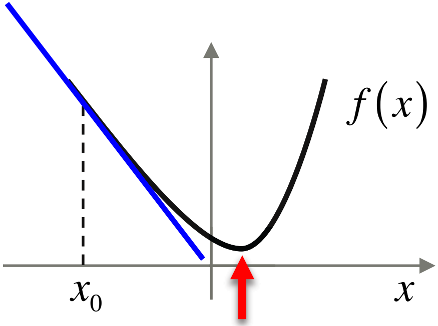
$$f\left(x_0 + \Delta x\right) \approx f\left(x_0\right) + J_f \Delta x + \varepsilon$$

$$\underset{\Delta x}{\text{minimize}} \quad \left\| b - f\left(x_0 + \Delta x\right) \right\|_2^2$$

$$\approx \left\| b - \left(f\left(x_0\right) + J_f \Delta x \right) \right\|_2^2$$

Review: Optimization

- find minimum of linear function approximation (gradient=0)



$$f(x_0 + \Delta x) \approx f(x_0) + J_f \Delta x + \varepsilon$$

$$\underset{\Delta x}{\text{minimize}} \quad \left\| b - f(x_0 + \Delta x) \right\|_2^2$$

$$\approx \left\| b - \left(f(x_0) + J_f \Delta x \right) \right\|_2^2$$

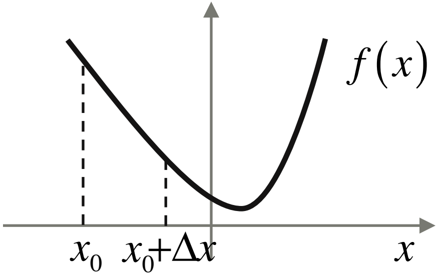
equate gradient to zero:

$$0 = J_f^T J_f \Delta x - J_f^T \left(b - f(x) \right) \quad \Rightarrow \quad \Delta x = \left(J_f^T J_f \right)^{-1} J_f^T \left(b - f(x) \right)$$

normal equations

Review: Optimization

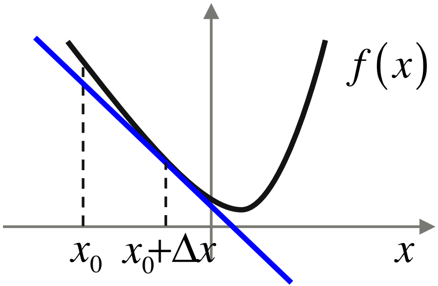
- take step and repeat procedure



$$\Delta x = \left(J_f^T J_f\right)^{-1} J_f^T \left(b - f(x)\right)$$

Review: Optimization

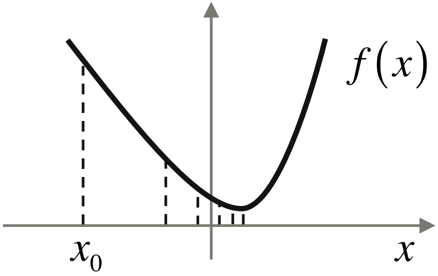
- take step and repeat procedure, will get there eventually



$$\Delta x = \left(J_f^T J_f\right)^{-1} J_f^T \left(b - f(x)\right)$$

Review: Optimization – Gauss-Newton

- results in an iterative algorithm

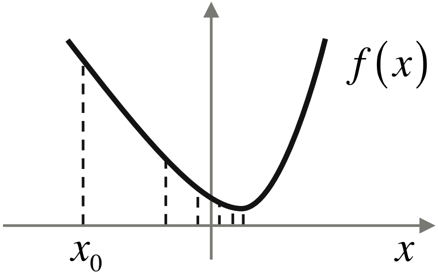


$$x_{k+1} = x_k + \Delta x$$

$$\Delta x = \left(J_f^T J_f\right)^{-1} J_f^T \left(b - f(x)\right)$$

Review: Optimization – Gauss-Newton

- results in an iterative algorithm

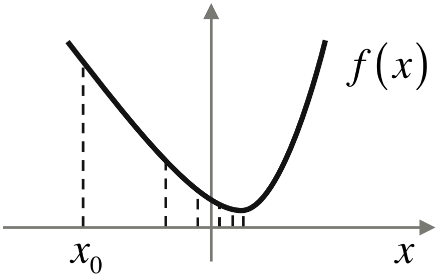



```
1. x = rand() // initialize x0
2. for k=1 to max_iter
3.   f = eval_objective(x)
4.   J = eval_jacobian(x)
5.   x = x + inv(J'*J)*J'*(b-f) // update x
```

$$\Delta x = \left(J_f^T J_f\right)^{-1} J_f^T \left(b - f(x)\right)$$

Review: Optimization – Gauss-Newton

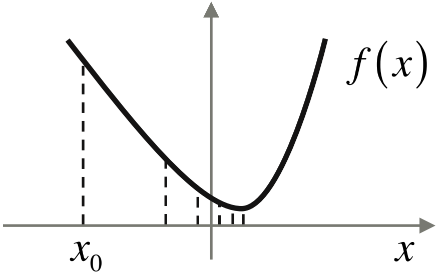
- matrix $J^T J$ can be ill-conditioned (i.e. not invertible)



$$\Delta x = \left(J_f^T J_f\right)^{-1} J_f^T \left(b - f(x)\right)$$

Review: Optimization – Levenberg

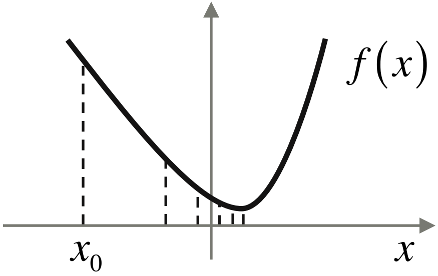
- matrix $J^T J$ can be ill-conditioned (i.e. not invertible)
- add a diagonal matrix to make invertible – acts as damping



$$\Delta x = \left(J_f^T J_f + \lambda I\right)^{-1} J_f^T \left(b - f(x)\right)$$

Review: Optimization – Levenberg-Marquardt

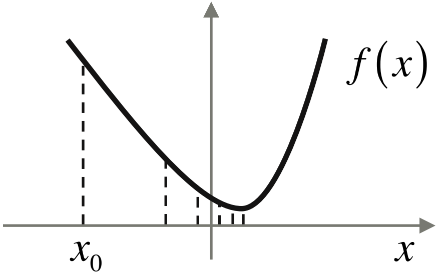
- matrix $J^T J$ can be ill-conditioned (i.e. not invertible)
- better: use $J^T J$ instead of I as damping. This is LM!



$$\Delta x = \left(J_f^T J_f + \lambda \, \mathrm{diag}\left(J_f^T J_f \right) \right)^{-1} J_f^T \left(b - f(x) \right)$$

Review: Optimization – Levenberg-Marquardt

- matrix $J^T J$ can be ill-conditioned (i.e. not invertible)
- better: use $J^T J$ instead of I as damping. This is LM!




```
x = rand() // initialize x0
for k=1 to max_iter
  f = eval_objective(x)
  J = eval_jacobian(x)
  x = x+inv(J'*J+lambda*diag(J'*J))*J'*(b-f)
```

$$\Delta x = \left(J_f^T J_f + \lambda \,\mathrm{diag}\left(J_f^T J_f\right)\right)^{-1} J_f^T \left(b - f(x)\right)$$

Pose Estimation via Levenberg-Marquardt

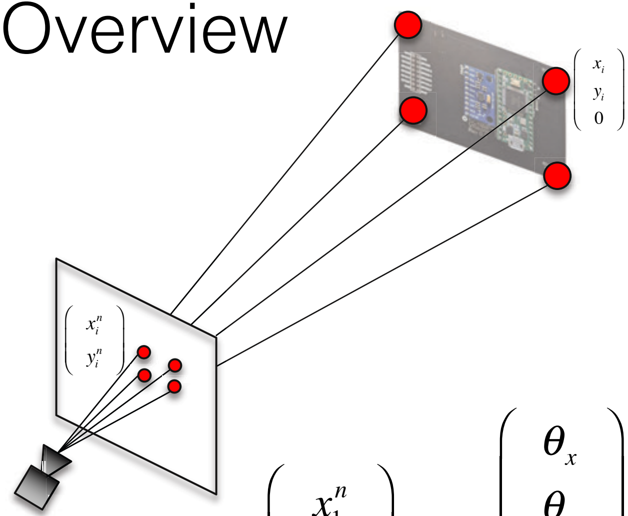
Pose Estimation - Overview

- goal: estimate pose via nonlinear least squares optimization

$$\underset{\{p\}}{\text{minimize}} \left\| b - f\left(g\left(p\right)\right) \right\|_2^2$$

↑ image formation

- minimize reprojection error
- pose *p* is 6-element vector with 3 Euler angles and translation of VRduino w.r.t. base station



$$b = \begin{pmatrix} x_1^n \\ y_1^n \\ \vdots \\ x_4^n \\ y_4^n \end{pmatrix} \qquad p = \begin{pmatrix} \theta_x \\ \theta_y \\ \theta_z \\ t_x \\ t_y \\ t_z \end{pmatrix}$$

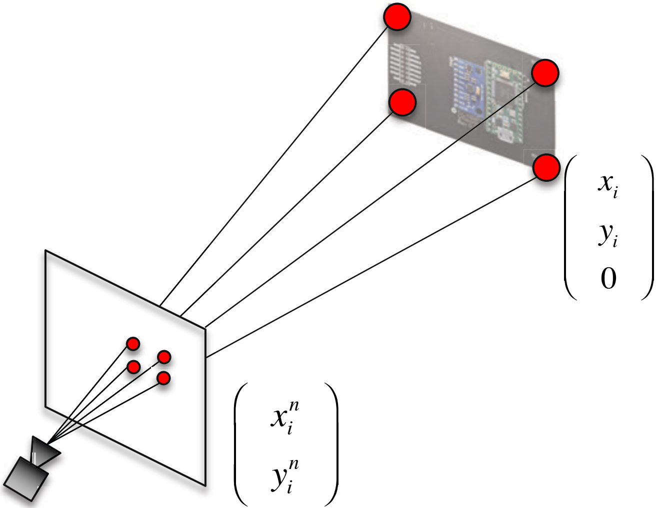
Pose Estimation - Objective Function

- goal: estimate pose via nonlinear least squares optimization

$$\underset{\{p\}}{\text{minimize}} \left\| b - f\left(g\left(p\right)\right) \right\|_2^2$$

↑

image formation



- objective function is sum of squares of reprojection error

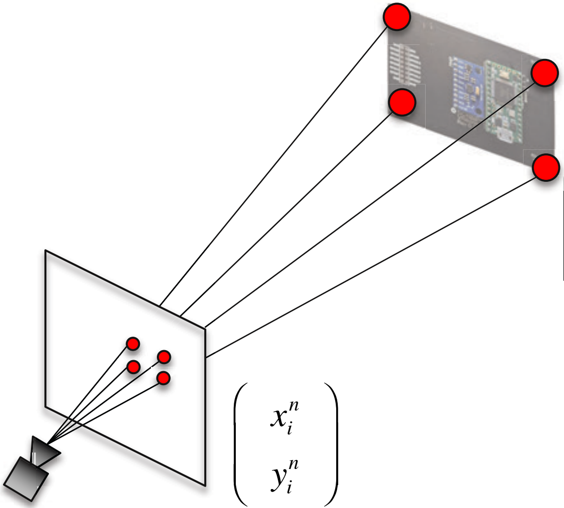
$$\left\| b - f\left(g\left(p\right)\right) \right\|_2^2 = \left(x_1^n - f_1\left(g\left(p\right)\right)\right)^2 + \left(y_1^n - f_2\left(g\left(p\right)\right)\right)^2 + \ldots + \left(x_4^n - f_7\left(g\left(p\right)\right)\right)^2 + \left(y_4^n - f_8\left(g\left(p\right)\right)\right)^2$$

Image Formation

1. transform 3D point into view space:

$$\begin{pmatrix} x_i^c \\ y_i^c \\ w_i^c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{pmatrix} \cdot \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix} \cdot \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}$$



$$\begin{pmatrix} x_i \\ y_i \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_i^n \\ y_i^n \end{pmatrix}$$

2. perspective divide:

$$\begin{pmatrix} x_i^n \\ y_i^n \end{pmatrix} = \begin{pmatrix} \frac{x_i^c}{w_i^c} \\ \frac{y_i^c}{w_i^c} \end{pmatrix}$$

Image Formation: *g(p)* and *f(h)*

- split up image formation into two functions

$$f(h) = f\left(g(p)\right)$$

$$g : \Re^6 \rightarrow \Re^9, \quad f : \Re^9 \rightarrow \Re^8$$

Image Formation: *f(h)*

- *f(h)* uses elements of homography matrix *h* to compute projected 2D coordinates as

$$
f(h) = \begin{pmatrix} f_1(h) \\ f_2(h) \\ \vdots \\ f_7(h) \\ f_8(h) \end{pmatrix} = \begin{pmatrix} x_1^n \\ y_1^n \\ \vdots \\ x_4^n \\ y_4^n \end{pmatrix} = \begin{pmatrix} \dfrac{h_1 x_1 + h_2 y_1 + h_3}{h_7 x_1 + h_8 y_1 + h_9} \\ \dfrac{h_4 x_1 + h_5 y_1 + h_6}{h_7 x_1 + h_8 y_1 + h_9} \\ \vdots \\ \dfrac{h_1 x_4 + h_2 y_4 + h_3}{h_7 x_4 + h_8 y_4 + h_9} \\ \dfrac{h_4 x_4 + h_5 y_4 + h_6}{h_7 x_4 + h_8 y_4 + h_9} \end{pmatrix}
$$

Jacobian Matrix of *f(h)*

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial h_1} & \cdots & \frac{\partial f_1}{\partial h_9} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_8}{\partial h_1} & \cdots & \frac{\partial f_8}{\partial h_9} \end{pmatrix}$$

$$f_1(h) = \frac{h_1\ x_1 + h_2\ y_1 + h_3}{h_7\ x_1 + h_8\ y_1 + h_9}$$

- first row of Jacobian matrix

$$\frac{\partial f_1}{\partial h_1} = \frac{x_1}{h_7\ x_1 + h_8\ y_1 + h_9}$$

$$\frac{\partial f_1}{\partial h_2} = \frac{y_1}{h_7\ x_1 + h_8\ y_1 + h_9}$$

$$\frac{\partial f_1}{\partial h_3} = \frac{1}{h_7\ x_1 + h_8\ y_1 + h_9}$$

$$\frac{\partial f_1}{\partial h_4} = 0, \quad \frac{\partial f_1}{\partial h_5} = 0, \quad \frac{\partial f_1}{\partial h_6} = 0$$

$$\frac{\partial f_1}{\partial h_7} = -\left(\frac{h_1\ x_1 + h_2\ y_1 + h_3}{(h_7\ x_1 + h_8\ y_1 + h_9)^2} \right) x_1$$

$$\frac{\partial f_1}{\partial h_8} = -\left(\frac{h_1\ x_1 + h_2\ y_1 + h_3}{(h_7\ x_1 + h_8\ y_1 + h_9)^2} \right) y_1$$

$$\frac{\partial f_1}{\partial h_9} = -\frac{h_1\ x_1 + h_2\ y_1 + h_3}{(h_7\ x_1 + h_8\ y_1 + h_9)^2}$$

Jacobian Matrix of *f(h)*

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial h_1} & \cdots & \frac{\partial f_1}{\partial h_9} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_8}{\partial h_1} & \cdots & \frac{\partial f_8}{\partial h_9} \end{pmatrix}$$

- the remaining rows of the Jacobian can be derived with a similar pattern

- see *course notes* for a detailed deriavtion of the elements of this Jacobian matrix

Image Formation: *g(p)*

- *g(p)* uses 6 pose parameters to compute elements of homography matrix *h as*

$$g(p)=\begin{pmatrix} g_1(p) \\ \vdots \\ g_9(p) \end{pmatrix}=\begin{pmatrix} h_1 \\ \vdots \\ h_9 \end{pmatrix}$$

definition of homography matrix:

$$\begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\cdot\begin{pmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ r_{31} & r_{32} & t_z \end{pmatrix}$$

rotation matrix from Euler angles:

$$R=R_z(\theta_z)\cdot R_x(\theta_x)\cdot R_y(\theta_y)$$

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}=\begin{pmatrix} \cos(\theta_z) & -\sin(\theta_z) & 0 \\ \sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & -\sin(\theta_x) \\ 0 & \sin(\theta_x) & \cos(\theta_x) \end{pmatrix}\begin{pmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta_y) \end{pmatrix}$$

Image Formation: *g(p)*

- write as

$$
\begin{aligned}
h_1 = g_1(p) &= \cos(\theta_y)\cos(\theta_z) - \sin(\theta_x)\sin(\theta_y)\sin(\theta_z) \\
h_2 = g_2(p) &= -\cos(\theta_x)\sin(\theta_z) \\
h_3 = g_3(p) &= t_x \\
h_4 = g_4(p) &= \cos(\theta_y)\sin(\theta_z) + \sin(\theta_x)\sin(\theta_y)\cos(\theta_z) \\
h_5 = g_5(p) &= \cos(\theta_x)\cos(\theta_z) \\
h_6 = g_6(p) &= t_y \\
h_7 = g_7(p) &= \cos(\theta_x)\sin(\theta_y) \\
h_8 = g_8(p) &= -\sin(\theta_x) \\
h_9 = g_9(p) &= -t_z
\end{aligned}
$$

Jacobian Matrix of *g(p)*

$$J_g = \begin{pmatrix} \frac{\partial g_1}{\partial p_1} & \cdots & \frac{\partial g_1}{\partial p_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_9}{\partial p_1} & \cdots & \frac{\partial g_9}{\partial p_6} \end{pmatrix}$$

$$p = \left(p_1, p_2, p_3, p_4, p_5, p_6\right) = \left(\theta_x, \theta_y, \theta_z, t_x, t_y, t_z\right)$$

$$\begin{aligned}
h_1 = g_1(p) &= \cos(\theta_y)\cos(\theta_z) - \sin(\theta_x)\sin(\theta_y)\sin(\theta_z) \\
h_2 = g_2(p) &= -\cos(\theta_x)\sin(\theta_z) \\
h_3 = g_3(p) &= t_x \\
h_4 = g_4(p) &= \cos(\theta_y)\sin(\theta_z) + \sin(\theta_x)\sin(\theta_y)\cos(\theta_z) \\
h_5 = g_5(p) &= \cos(\theta_x)\cos(\theta_z) \\
h_6 = g_6(p) &= t_y \\
h_7 = g_7(p) &= \cos(\theta_x)\sin(\theta_y) \\
h_8 = g_8(p) &= -\sin(\theta_x) \\
h_9 = g_9(p) &= -t_z
\end{aligned}$$

Jacobian Matrix of *g(p)*

$$J_g = \begin{pmatrix} \frac{\partial g_1}{\partial p_1} & \cdots & \frac{\partial g_1}{\partial p_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_9}{\partial p_1} & \cdots & \frac{\partial g_9}{\partial p_6} \end{pmatrix}$$

$$p = \left(p_1, p_2, p_3, p_4, p_5, p_6\right) = \left(\theta_x, \theta_y, \theta_z, t_x, t_y, t_z\right)$$

$$h_1 = g_1(p) = \cos(\theta_y)\cos(\theta_z) - \sin(\theta_x)\sin(\theta_y)\sin(\theta_z)$$

- first row of Jacobian matrix

$$\frac{\partial g_1}{\partial p_1} = -\cos(\theta_x)\sin(\theta_y)\sin(\theta_z)$$

$$\frac{\partial g_1}{\partial p_2} = -\sin(\theta_y)\cos(\theta_z) - \sin(\theta_x)\cos(\theta_y)\sin(\theta_z)$$

$$\frac{\partial g_1}{\partial p_3} = -\cos(\theta_y)\sin(\theta_z) - \sin(\theta_x)\sin(\theta_y)\cos(\theta_z)$$

$$\frac{\partial g_1}{\partial p_4} = 0, \quad \frac{\partial g_1}{\partial p_5} = 0, \quad \frac{\partial g_1}{\partial p_6} = 0$$

Jacobian Matrix of *g(p)*

$$J_g = \begin{pmatrix} \frac{\partial g_1}{\partial p_1} & \cdots & \frac{\partial g_1}{\partial p_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_9}{\partial p_1} & \cdots & \frac{\partial g_9}{\partial p_6} \end{pmatrix}$$

- the remaining rows of the Jacobian can be derived with a similar pattern

- see *course notes* for a detailed deriavtion of the elements of this Jacobian matrix

Jacobian Matrices of *f* and *g*

- to get the Jacobian of *f(g(p))*, compute the two Jacobian matrices and multiply them

$$J = J_f \cdot J_g = \begin{pmatrix} \frac{\partial f_1}{\partial h_1} & \cdots & \frac{\partial f_1}{\partial h_9} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_8}{\partial h_1} & \cdots & \frac{\partial f_8}{\partial h_9} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial g_1}{\partial p_1} & \cdots & \frac{\partial g_1}{\partial p_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_9}{\partial p_1} & \cdots & \frac{\partial g_9}{\partial p_6} \end{pmatrix}$$

Pose Tracking with LM

- LM then iteratively updates pose as

$$p^{(k+1)} = p^{(k)} + \left(J^T J + \lambda\,\mathrm{diag}\left(J^T J\right)\right)^{-1} J^T \left(b - f\left(g\left(p^{(k)}\right)\right)\right)$$

- pseudo-code

```
1. p = ... // initialize p0
2. for k=1 to max_iter
3.   f = eval_objective(p)
4.   J = get_jacobian(p)
5.   p = p + inv(J'*J+lambda*diag(J'*J))*J'*(b-f)
```

Pose Tracking with LM

```
value = function eval_objective(p)
  for i=1:4
    value(2*(i-1)) = ...
    value(2*(i-1)+1) = ...
```

$$\frac{h_1 x_i + h_2 y_i + h_3}{h_7 x_i + h_8 y_i + h_9}$$

$$\frac{h_4 x_i + h_5 y_i + h_6}{h_7 x_i + h_8 y_i + h_9}$$

Pose Tracking with LM

```
1. J = function get_jacobian(p)
2.    Jf = get_jacobian_f(g(p))
3.    Jg = get_jacobian_g(p)
4.    J = Jf*Jg
```

$$J_f = \begin{pmatrix} \frac{\partial f_1}{\partial h_1} & \cdots & \frac{\partial f_1}{\partial h_9} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_8}{\partial h_1} & \cdots & \frac{\partial f_8}{\partial h_9} \end{pmatrix}$$

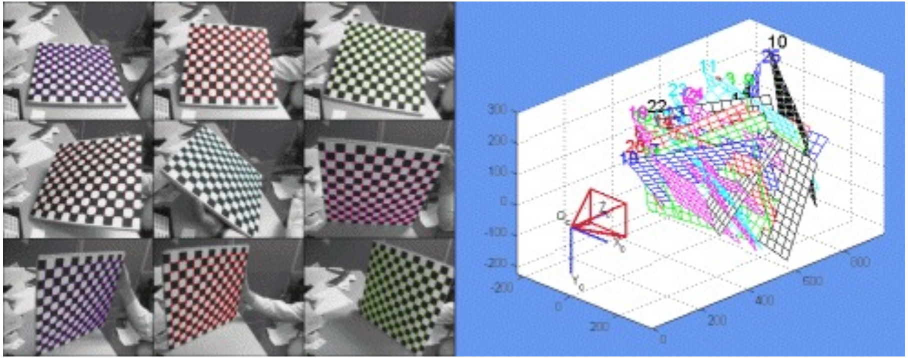
$$J_g = \begin{pmatrix} \frac{\partial g_1}{\partial p_1} & \cdots & \frac{\partial g_1}{\partial p_6} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_9}{\partial p_1} & \cdots & \frac{\partial g_9}{\partial p_6} \end{pmatrix}$$

Pose Tracking with LM on VRduino

- some more hints for implementation:
  - let Arduino Matrix library compute matrix-matrix multiplications and also matrix inverses for you!
  - run homography method and use that to initialize p for LM
  - use something like 5-25 interations of LM per frame for real-time performance
  - user-defined parameter λ
  - good luck!

Outlook: Camera Calibration

- camera calibration is one of the most fundamental problems in computer vision and imaging
- task: estimate intrinsic (lens distortion, focal length, principle point) & extrinsic (translation, rotation) camera parameters given images of planar checkerboards
- uses similar procedure as discussed today



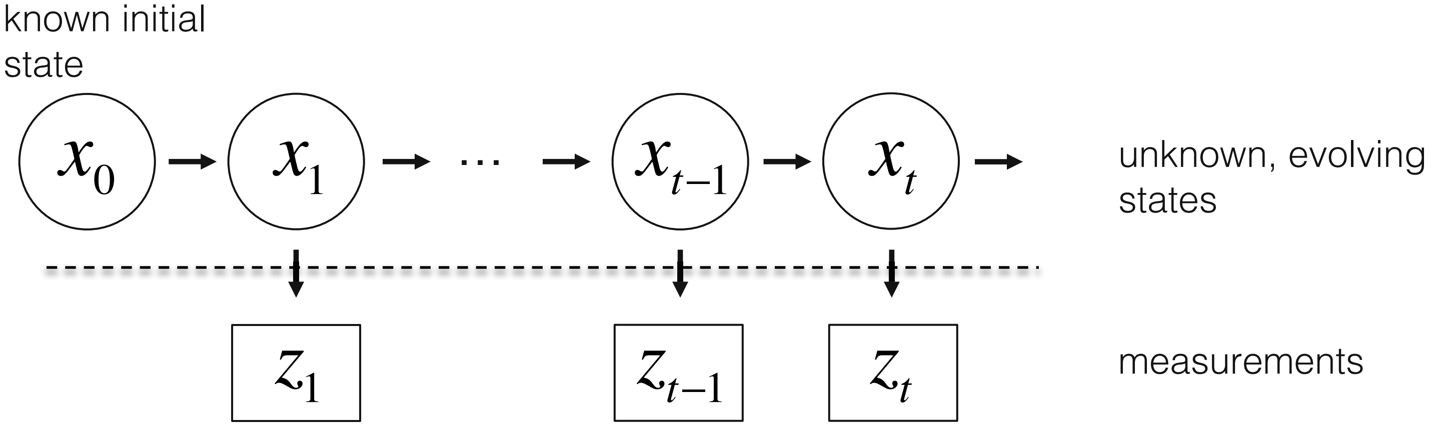
http://www.vision.caltech.edu/bouguetj/calib_doc/

Outlook: Sensor Fusion with Extended Kalman Filter

- also desirable: estimate bias of each of all IMU sensors
- also desirable: joint pose estimation from all IMU + photodiode measurements
- can do all of that with an Extended Kalman Filter - slightly too advanced for this class, but you can find a lot of literature in the robotic vision community

Outlook: Sensor Fusion with Extended Kalman Filter

- Extended Kalman filter: can be interpreted as a Bayesian framework for sensor fusion
- Hidden Markov Model (HMM)



Must read: course notes on tracking!