Albrecht Dürer, “Underweysung der Messung mit dem Zirckel und Richtscheyt”, 1525
Lecture Overview

• what is computer graphics?
• the graphics pipeline
• primitives: vertices, edges, triangles!
• model transforms: translations, rotations, scaling
• view transform
• perspective transform
• window transform
Modeling 3D Geometry

Courtesy of H.G. Animations
https://www.youtube.com/watch?v=fewbFvA5oGk
What is Computer Graphics?

- at the most basic level: conversion from 3D scene description to 2D image

- what do you need to describe a static scene?
  - 3D geometry and transformations
  - lights
  - material properties

- most common geometry primitives in graphics:
  - vertices (3D points) and normals (unit-length vector associated with vertex)
  - triangles (set of 3 vertices, high-resolution 3D models have M or B of triangles)
The Graphics Pipeline

- geometry + transformations
- cameras and viewing
- lighting and shading
- rasterization
- texturing
Some History

- Stanford startup in 1981
- computer graphics goes hardware
- based on Jim Clark’s geometry engine
Some History

The subsystems are:

- **Matrix Subsystem** - A stack of 4x4 floating-point matrices for completely general, 2D or 3D floating-point coordinate transformation of graphical data.

- **Clipping Subsystem** - A windowing, or clipping, capability for clipping 2D or 3D graphical data to a window into the user's virtual drawing space. In 3D, this window is a volume of the user's virtual, floating point space, corresponding to a truncated viewing pyramid with "near" and "far" clipping.

- **Scaling Subsystem** - Scaling of 2D and 3D coordinates to the coordinate system of the particular output device of the user. In 3D, this scaling phase also includes either orthographic or perspective projection onto the viewer's virtual window. Stereo coordinates are computed and optionally supplied as the output of the system.
The Graphics Pipeline

- monolithic graphics workstations of the 80s have been replaced by modular GPUs (graphics processing units); major companies: NVIDIA, AMD, Intel

- early versions of these GPUs implemented **fixed-function** rendering pipeline in hardware

- GPUs have become programmable starting in the late 90s
  - e.g. in 2001 Nvidia GeForce 3 = first programmable shaders

- now: GPUs = programmable (e.g. OpenGL, CUDA, OpenCL) processors
GPU = massively parallel processor
• OpenGL is our interface to the GPU!

• right: “old-school” OpenGL state machine

• today’s lecture: vertex transforms
WebGL

- JavaScript application programmer interface (API) for 2D and 3D graphics

- OpenGL ES 2.0 running in the browser, implemented by all modern browsers

- overview, tutorials, documentation: see lab 1
three.js

• cross-browser JavaScript library/API

• higher-level library that provides a lot of useful helper functions, tools, and abstractions around WebGL – easy and convenient to use

• https://threejs.org/
• simple examples: https://threejs.org/examples/

• great introduction (in WebGL):
  http://davidscottlyons.com/threejs/presentations/frontporch14/
The Graphics Pipeline

3D Graphics Rendering Pipeline: Output of one stage is fed as input of the next stage. A vertex has attributes such as \((x, y, z)\) position, color (RGB or RGBA), vertex-normal \((n_x, n_y, n_z)\), and texture. A primitive is made up of one or more vertices. The rasterizer raster-scans each primitive to produce a set of grid-aligned fragments, by interpolating the vertices.
3D Geometry: Input to Graphics Pipeline

Courtesy of H.G. Animations
https://www.youtube.com/watch?v=fewbFvA5oGk
Primitives

- vertex = 3D point \( v(x,y,z) \)
- triangle = 3 vertices
- normal = 3D vector per vertex describing surface orientation \( \mathbf{n} = (n_x, n_y, n_z) \)

2. Rasterization: Convert each primitive (connected vertices) into a set of fragments. A fragment can be treated as a pixel in 3D spaces, which is aligned with the pixel grid, with attributes such as position, color, normal and texture.

3. Fragment Processing: Process individual fragments.

4. Output Merging: Combine the fragments of all primitives (in 3D space) into 2D color-pixel for the display.
Pixels v Fragments

- fragments have rasterized 2D coordinates on screen but a lot of other attributes too (texture coordinates, depth value, alpha value, …)
- pixels appear on screen
- won’t discuss in more detail today
1. **Vertex Processing**: Process and transform individual vertices & normals.

2. **Rasterization**: Convert each primitive (connected vertices) into a set of fragments. A fragment can be treated as a pixel in 3D spaces, which is aligned with the pixel grid, with attributes such as position, color, normal, and texture.

3. **Fragment Processing**: Process individual fragments.

4. **Output Merging**: Combine the fragments of all primitives (in 3D space) into 2D color-pixel for the display.

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https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html
The Graphics Pipeline

vertex shader

- transforms & (per-vertex) lighting

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The Graphics Pipeline

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The Graphics Pipeline

Vertex Shader

Rasterizer

Fragment Processor (Programmable)

Output Merging

Pixels

Display

Raw Vertices & Primitives

Transformed Vertices & Primitives

Fragments

Processed Fragments

3D

2D array of color-values

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The Graphics Pipeline

- Fragment shader
  - Texturing
  - (Per-fragment) lighting

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The Graphics Pipeline

- Raw Vertices & Primitives
- Transformed Vertices & Primitives
- Rasterizer
- Fragments
- Processed Fragments
- Output Merging

Vertex Processor (Programmable)

Fragment Processor (Programmable)

Display

3D

2D array of color-values

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The Graphics Pipeline

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Coordinate Systems

- right hand coordinate system

- a few different coordinate systems:
  - object coordinates
  - world coordinates
  - viewing coordinates
  - also clip, normalized device, and window coordinates
Vertex Transforms

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Vertex Transforms

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1. Arrange the objects (or models, or avatar) in the world (Model Transform).
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1. Arrange the objects (or models, or avatar) in the world (Model Transform).
2. Position and orientation the camera (View transform).
3. Select a camera lens (wide angle, normal or telescopic), adjust the focus length and zoom factor to set the camera's field of view (Projection transform).

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Model Transform

- transform each vertex \( v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) from object coordinates to world coordinates

https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html
Model Transform - Scaling

- transform each vertex $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ from object coordinates to world coordinates

1. scaling as 3x3 matrix

$$S(s_x, s_y, s_z) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix}$$

scaled vertex = matrix-vector product:

$$Sv = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \\ s_z z \end{pmatrix}$$
Model Transform - Rotation

• transform each vertex $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ from object coordinates to world coordinates

2. rotation as 3x3 matrix

$$R_z(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

rotated vertex = matrix-vector product, e.g.

$$R_zv = \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \\ z \end{pmatrix}$$
Model Transform - Translation

- transform each vertex \( v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \) from object coordinates to world coordinates

3. translation cannot be represented as 3x3 matrix!

\[
\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \begin{pmatrix} x + d_x \\ y + d_y \\ z + d_z \end{pmatrix}
\]

that's unfortunate 😔
Model Transform - Translation

- solution: use homogeneous coordinates, vertex is $v = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$

3. translation is 4x4 matrix

$$T(d) = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$Tv = \begin{pmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x + d_x \\ y + d_y \\ z + d_z \\ 1 \end{pmatrix}$$

better 😊
Summary of Homogeneous Matrix Transforms

- **translation** \( T(d) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

- **scale** \( S(s) = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

- **rotation** \( R_z(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

\[ R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ R_y = \begin{bmatrix} \cos \theta & 0 & \sin \theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin \theta & 0 & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

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**Inverse Transforms**

- **translation** \( T^{-1}(d) = T(-d) = \begin{bmatrix} 1 & 0 & 0 & -d_x \\ 0 & 1 & 0 & -d_y \\ 0 & 0 & 1 & -d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

- **scale** \( S^{-1}(s) = S\left(\frac{1}{s}\right) = \begin{bmatrix} 1/s_x & 0 & 0 & 0 \\ 0 & 1/s_y & 0 & 0 \\ 0 & 0 & 1/s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

- **rotation** \( R_z^{-1}(\theta) = R_z(-\theta) = \begin{bmatrix} \cos -\theta & -\sin -\theta & 0 & 0 \\ \sin -\theta & \cos -\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \)

Read more: [https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html](https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html)
Summary of Homogeneous Matrix Transforms

- successive transforms: $\mathbf{v}' = T \cdot S \cdot R_z \cdot R_x \cdot T \cdot \mathbf{v}$

- inverse successive transforms: 
  \[
  \mathbf{v} = \left( T \cdot S \cdot R_z \cdot R_x \cdot T \right)^{-1} \cdot \mathbf{v}' \\
  = T^{-1} \cdot R_x^{-1} \cdot R_z^{-1} \cdot S^{-1} \cdot T^{-1} \cdot \mathbf{v}'
  \]

Read more: https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html
Vector and Normal Transforms

- homogeneous representation of a vector $t$, i.e. pointing from $v_1$ to $v_2$:

$$t = \begin{pmatrix} (v_2 - v_1)_x \\ (v_2 - v_1)_y \\ (v_2 - v_1)_z \\ (1-1) \end{pmatrix} = \begin{pmatrix} t_x \\ t_y \\ t_z \\ 0 \end{pmatrix}$$

- successive transforms: $t' = M \cdot t = M \cdot (v_2 - v_1) = M \cdot v_2 - M \cdot v_1$

- this works!
Vector and Normal Transforms

- homogeneous representation of a normal (unit length, perpendicular to surface)
- Transforms:
  - Translation \( n' = T \cdot n \quad (n' = n) \)
  - Rotation
  - Scaling
Vector and **Normal** Transforms

- homogeneous representation of a normal (unit length, perpendicular to surface)

\[ n = \begin{pmatrix}
  n_x \\
  n_y \\
  n_z \\
  0
\end{pmatrix} \]

- Transforms:
  - Translation \( n' = T \cdot n \) \((n' = n)\)
  - Rotation \( n' = R \cdot n \)
  - Scaling
Vector and **Normal Transforms**

- **homogeneous representation of a normal**
  (unit length, perpendicular to surface)

\[
\begin{pmatrix}
    n_x \\
    n_y \\
    n_z \\
    0
\end{pmatrix}
\]

- **Transforms:**
  - **Translation** \( n' = T \cdot n \) \((n' = n)\)
  - **Rotation** \( n' = R \cdot n \)
  - **Scaling** \( n' = S \cdot n \)
Vector and **Normal** Transforms

- homogeneous representation of a normal (unit length, perpendicular to surface)

\[
\begin{pmatrix}
n_x \\
n_y \\
n_z \\
0
\end{pmatrix}
\]

- Transforms:
  - Translation \( n' = T \cdot n \) \((n' = n)\)
  - Rotation \( n' = R \cdot n \)
  - Scaling \( n' = S \cdot n \)

Non-uniform scaling breaks orthogonality

Does not work!
Vector and Normal Transforms

- homogeneous representation of a normal (unit length, perpendicular to surface)

- need to use normal matrix = transpose of inverse for transformation!

\[ n = \begin{pmatrix} n_x \\ n_y \\ n_z \\ 0 \end{pmatrix} \]

\[ n' = (M^{-1})^T \cdot n \]
Vector and **Normal** Transforms

- homogeneous representation of a normal (unit length, perpendicular to surface)

\[
n = \begin{pmatrix} n_x \\ n_y \\ n_z \\ 0 \end{pmatrix}
\]

- need to use normal matrix = transpose of inverse for transformation!

\[
n' = \left( M^{-1} \right)^T \cdot n
\]

- fine print: only use upper left 3x3 part of modelview matrix for inverse transpose (no homogeneous normal representation) OR drop \( w \) component from \( n' \) after multiplying 4x4 inverse transpose (i.e. don’t use \( w \) for normalization of \( n' \! )

Attention!

- rotations and translations (or transforms in general) are not commutative!
- make sure you get the correct order!
so far we discussed model transforms, e.g. going from object or model space to world space
so far we discussed model transforms, e.g. going from object or model space to world space
one simple 4x4 transform matrix is sufficient to go from world space to camera or view space!

https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html
View Transform

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View Transform

specify camera by

- **eye position** \( \text{eye} = \begin{pmatrix} \text{eye}_x \\ \text{eye}_y \\ \text{eye}_z \end{pmatrix} \)

- **reference position** \( \text{center} = \begin{pmatrix} \text{center}_x \\ \text{center}_y \\ \text{center}_z \end{pmatrix} \)

- **up vector** \( \text{up} = \begin{pmatrix} \text{up}_x \\ \text{up}_y \\ \text{up}_z \end{pmatrix} \)
View Transform

specify camera by

- **eye position**
  \[
  \text{eye} = \begin{pmatrix}
  \text{eye}_x \\
  \text{eye}_y \\
  \text{eye}_z
  \end{pmatrix}
  \]

- **reference position**
  \[
  \text{center} = \begin{pmatrix}
  \text{center}_x \\
  \text{center}_y \\
  \text{center}_z
  \end{pmatrix}
  \]

- **up vector**
  \[
  \text{up} = \begin{pmatrix}
  \text{up}_x \\
  \text{up}_y \\
  \text{up}_z
  \end{pmatrix}
  \]

compute 3 vectors:

\[
\text{z}^c = \frac{\text{eye} - \text{center}}{\|\text{eye} - \text{center}\|}
\]

\[
\text{x}^c = \frac{\text{up} \times \text{z}^c}{\|\text{up} \times \text{z}^c\|}
\]

\[
\text{y}^c = \text{z}^c \times \text{x}^c
\]
View Transform

view transform $M$ is translation into eye position, followed by rotation

compute 3 vectors:

\[ z^c = \frac{\text{eye} - \text{center}}{||\text{eye} - \text{center}||} \]

\[ x^c = \frac{\text{up} \times z^c}{||\text{up} \times z^c||} \]

\[ y^c = z^c \times x^c \]
view transform $M$ is translation into eye position, followed by rotation

compute 3 vectors:

$z^c = \frac{\text{eye} - \text{center}}{||\text{eye} - \text{center}||}$

$x^c = \frac{\text{up} \times z^c}{||\text{up} \times z^c||}$

$y^c = z^c \times x^c$

$M = R \cdot T(-e) = \begin{pmatrix} x_x^c & x_y^c & x_z^c & 0 \\ y_x^c & y_y^c & y_z^c & 0 \\ z_x^c & z_y^c & z_z^c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\text{eye}_x \\ 0 & 1 & 0 & -\text{eye}_y \\ 0 & 0 & 1 & -\text{eye}_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$
view transform $M$ is translation into eye position, followed by rotation

$$M = R \cdot T(-e) = \begin{pmatrix}
x_x^c & x_y^c & x_z^c & -(x_x^c eye_x + x_y^c eye_y + x_z^c eye_z) \\
y_x^c & y_y^c & y_z^c & -(y_x^c eye_x + y_y^c eye_y + y_z^c eye_z) \\
z_x^c & z_y^c & z_z^c & -(z_x^c eye_x + z_y^c eye_y + z_z^c eye_z) \\
0 & 0 & 0 & 1
\end{pmatrix}$$
many graphics APIs have a function called `lookat` that automatically computes the view matrix for you.

Three.js also has such a function, but that only computes the rotation, not the translation, of the view matrix. So best implement the view matrix yourself!
View Transform

- in camera/view space, the camera is at the origin, looking into negative z
- *modelview matrix* is combined model (rotations, translations, scaling) and view matrix!
View Transform

- in camera/view space, the camera is at the origin, looking into negative z
Projection Transform

• Just a 4x4 matrix projecting a 3D point onto a 2D canvas
• Similar to choosing lens and sensor of camera – specify field of view and aspect

https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html
Projection Transform - Perspective Projection

- have **symmetric** view frustum
- fovy: vertical angle in degrees
- aspect: ratio of width/height
- zNear: near clipping plane (relative from cam)
- zFar: far clipping plane (relative from cam)

\[ f = \cot\left(\frac{\text{fovy}}{2}\right) \]

The projection matrix for a symmetric frustum is given by:

\[
M_{\text{proj}} = \begin{bmatrix}
\frac{f}{\text{aspect}} & 0 & 0 & 0 \\
0 & f & 0 & 0 \\
0 & 0 & -\frac{zFar + zNear}{zFar - zNear} & \frac{2 \cdot zFar \cdot zNear}{zFar - zNear} \\
0 & 0 & -1 & 0
\end{bmatrix}
\]

projection matrix
(symetric frustum)
Projection Transform - Perspective Projection

more general: a perspective “frustum” (truncated, possibly sheared pyramid)

- left (l), right (r), bottom (b), top (t): corner coordinates on near clipping plane (at zNear)

\[
M_{\text{proj}} = \begin{bmatrix}
\frac{2 \cdot z\text{Near}}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\
0 & \frac{2 \cdot z\text{Near}}{t-b} & \frac{t+b}{t-b} & 0 \\
0 & 0 & -\frac{z\text{Far} + z\text{Near}}{z\text{Far} - z\text{Near}} & \frac{2 \cdot z\text{Far} \cdot z\text{Near}}{z\text{Far} - z\text{Near}} \\
0 & 0 & \frac{z\text{Far} - z\text{Near}}{z\text{Far} - z\text{Near}} & 0
\end{bmatrix}
\]
Projection Transform - Orthographic Projection

more general: a “box frustum” (no perspective, objects don’t get smaller when farther away)

- left (l), right (r), bottom (b), top (t): corner coordinates on near clipping plane

\[
M_{\text{proj}} = \begin{pmatrix}
\frac{2}{r-l} & 0 & 0 & -\frac{r+l}{r-l} \\
0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\
0 & 0 & \frac{-2}{f-n} & -\frac{f+n}{f-n} \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

projection matrix (orthographic)
Projection Transform

- possible source of confusion for zNear and zFar:
  - Marschner & Shirley define it as absolute z coordinates, thus $z_{\text{Near}} > z_{\text{Far}}$ and both values are always negative
  - OpenGL and we define it as positive values, i.e. the distances of the near and far clipping plane from the camera ($z_{\text{Far}} > z_{\text{Near}}$)
Modelview Projection Matrix

- put it all together with 4x4 matrix multiplications!

\[ v_{\text{clip}} = \mathbf{M}_{\text{proj}} \cdot \mathbf{M}_{\text{view}} \cdot \mathbf{M}_{\text{model}} \cdot \mathbf{v} = \mathbf{M}_{\text{proj}} \cdot \mathbf{M}_{\text{mv}} \cdot \mathbf{v} \]

vertex in clip space  
projection matrix  
modelview matrix
Clip Space

https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html
Normalized Device Coordinates (NDC)

- not shown in previous illustration
- get to NDC by perspective division

$$v_{clip} = \begin{pmatrix} x_{clip} \\ y_{clip} \\ z_{clip} \\ w_{clip} \end{pmatrix} \quad \rightarrow \quad v_{NDC} = \begin{pmatrix} x_{clip} / w_{clip} \\ y_{clip} / w_{clip} \\ z_{clip} / w_{clip} \\ 1 \end{pmatrix} \quad \epsilon (-1,1)$$

vertex in clip space \quad \rightarrow \quad \text{vertex in NDC}
Viewport Transform

define (sub)window as viewport(x, y, width, height),

- x, y lower left corner of viewport rectangle (default is (0,0))
- width, height size of viewport rectangle in pixels

\[
\begin{align*}
    x_{\text{window}} &= \frac{\text{width}}{2} \left( x_{\text{NDC}} + 1 \right) + x \\
    y_{\text{window}} &= \frac{\text{height}}{2} \left( y_{\text{NDC}} + 1 \right) + y \\
    z_{\text{window}} &= \frac{1}{2} z_{\text{NDC}} + \frac{1}{2}
\end{align*}
\]

\[
\begin{align*}
    v_{\text{NDC}} &= \begin{pmatrix} x_{\text{clip}} / w_{\text{clip}} \\ y_{\text{clip}} / w_{\text{clip}} \\ z_{\text{clip}} / w_{\text{clip}} \\ 1 \end{pmatrix} \\
    v_{\text{window}} &= \begin{pmatrix} x_{\text{window}} \\ y_{\text{window}} \\ z_{\text{window}} \end{pmatrix} \in (0, width - 1) \quad \in (0, height - 1) \quad \in (0, 1)
\end{align*}
\]

vertex in NDC \hspace{1cm} vertex in window coords
all vertex transforms from today!
... and we can almost do this ...
• assign fixed color (e.g. red) to each vertex in window coordinates (fragment)
• interpolate (i.e. rasterize) lines between vertices (as defined by user)
Summary

• graphics pipeline is a series of operations that takes 3D vertices/normals/triangles as input and generates fragments and pixels

• today, we only discussed a part of it: vertex and normal transforms

• transforms include: rotation, scale, translation, perspective projection, perspective division, and viewport transform

• most transforms are represented as 4x4 matrices in homogeneous coordinates → know your matrices & be able to create, manipulate, invert them!
Next Lecture: Lighting and Shading, Fragment Processing

- vertex shader
  - transforms & (per-vertex) lighting

- fragment shader
  - texturing
  - (per-fragment) lighting

https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html
Further Reading

• course notes on transforms (see course website)

• Very good computer graphics explanations: https://www.scratchapixel.com

• good overview of OpenGL (deprecated version) and graphics pipeline (missing a few things) : https://www.ntu.edu.sg/home/ehchua/programming/opengl/CG_BasicsTheory.html


• WebGL / three.js tutorials: https://threejs.org/