# EE276: Homework #1

Due on Friday Jan 16, 6pm - Gradescope entry code: E6VP4X

1. **Example of joint entropy.** Let $p(x, y)$ be given by

| $X \backslash Y$ | 0 | 1 |
|---|---|---|
| 0 | $\frac{1}{10}$ | $\frac{3}{10}$ |
| 1 | $\frac{2}{10}$ | $\frac{4}{10}$ |

   Find

   (a) $H(X), H(Y)$.
   (b) $H(X \mid Y), H(Y \mid X)$.
   (c) $H(X, Y)$.
   (d) $H(Y) - H(Y \mid X)$.
   (e) $I(X; Y)$.
   (f) Draw a Venn diagram relating the quantities in (a) through (e).

   Numerically round the answers to three decimal places.

2. **Entropy of functions of a random variable.** Let $X$ be a discrete random variable. Show that the entropy of a function of $X$ is less than or equal to the entropy of $X$ by justifying the following steps:

$$H(X, g(X)) \stackrel{(a)}{=} H(X) + H(g(X) \mid X) \quad (1)$$

$$\stackrel{(b)}{=} H(X); \quad (2)$$

$$H(X, g(X)) \stackrel{(c)}{=} H(g(X)) + H(X \mid g(X)) \quad (3)$$

$$\stackrel{(d)}{\geq} H(g(X)). \quad (4)$$

   Thus $H(g(X)) \leq H(X)$.

3. **Entropy of a disjoint mixture.** Let $X_1$ and $X_2$ be discrete random variables drawn according to probability mass functions $p_1(\cdot)$ and $p_2(\cdot)$ over the respective alphabets $\mathcal{X}_1 = \{1, 2, \ldots, m\}$ and $\mathcal{X}_2 = \{m+1, \ldots, n\}$. Let

$$X = \begin{cases} X_1, & \text{with probability } \alpha, \\ X_2, & \text{with probability } 1 - \alpha. \end{cases}$$

   (a) Find $H(X)$ in terms of $H(X_1)$ and $H(X_2)$ and $\alpha$.

(b) Maximize over $\alpha$ to show that $2^{H(X)} \le 2^{H(X_1)} + 2^{H(X_2)}$ and interpret using the notion that $2^{H(X)}$ is the effective alphabet size.

4. **Coin flips.** A fair coin is flipped until the first head occurs. Let $X$ denote the number of flips required.

   (a) Find the entropy $H(X)$ in bits. The following expressions may be useful:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad \sum_{n=0}^{\infty} n r^n = \frac{r}{(1-r)^2}.$$

   (b) A random variable $X$ is drawn according to this distribution. Construct an "efficient" sequence of yes-no questions of the form, "Is $X$ contained in the set $S$?" that determine the value of $X$. Compare $H(X)$ to the expected number of questions required to determine $X$.

5. **Minimum entropy.** In the following, we use $H(p_1, ..., p_n) \equiv H(\mathbf{p})$ to denote the entropy $H(X)$ of a random variable $X$ with alphabet $\mathcal{X} := \{1, \ldots, n\}$, i.e.,

$$H(X) = -\sum_{i=1}^{n} p_i \log(p_i).$$

   What is the minimum value of $H(p_1, ..., p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of $n$-dimensional probability vectors? Find all $\mathbf{p}$'s which achieve this minimum.

6. **Mixing increases entropy.** Let $p_i > 0, \quad i = 1, 2, \ldots m$. Show that the entropy of a random variable distributed according to $(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)$, is less than or equal to the entropy of a random variable distributed according to $(p_1, \ldots, \frac{p_i+p_j}{2}, \ldots, \frac{p_i+p_j}{2}, \ldots, p_m)$.

7. **Infinite entropy. [Bonus]**
   This problem shows that the entropy of a discrete random variable can be infinite. (In this question you can take log as the natural logarithm for simplicity.)

   (a) Let $A = \sum_{n=2}^{\infty} (n \log^2 n)^{-1}$. Show that $A$ is finite by bounding the infinite sum by the integral of $(x \log^2 x)^{-1}$.

   (b) Show that the integer-valued random variable $X$ distributed as: $P(X = n) = (An \log^2 n)^{-1}$ for $n = 2, 3, \ldots$ has entropy $H(X)$ given by:

$$H(X) = \log A + \sum_{n=2}^{\infty} \frac{1}{An \log n} + \sum_{n=2}^{\infty} \frac{2 \log \log n}{An \log^2 n}$$

   (c) Show that the entropy $H(X) = +\infty$ (by showing that the sum $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges).