EE276 Homework #5

Due on Friday Feb 20, 6pm - Gradescope entry code: E6VP4X

1. **Zero-error capacity.** A channel with alphabet $\{0, 1, 2, 3, 4\}$ has transition probabilities of the form

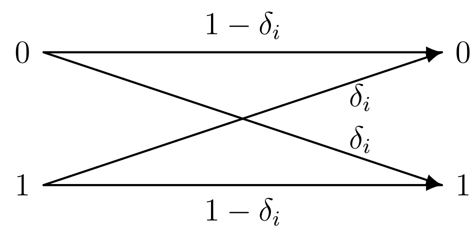
$$p(y|x) = \begin{cases} 1/2 & \text{if } y = x \pm 1 \bmod 5 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Compute the capacity of this channel in bits.

   (b) The zero-error capacity of a channel is the number of bits per channel use that can be transmitted with zero probability of error. Clearly, the zero-error capacity of this pentagonal channel is at least 1 bit (transmit 0 or 1 with probability 1/2). Find a block code that shows that the zero-error capacity is greater than 1 bit. Can you estimate the exact value of the zero-error capacity?

   (*Hint*: Consider codes of length 2 for this channel.)

2. **Time-varying channels.**
Consider a time-varying discrete *memoryless* channel. Let $Y_1, Y_2, \ldots, Y_n$ be conditionally independent given $X_1, X_2, \ldots, X_n$, with conditional distribution given by $p(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^{n} p_i(y_i|x_i)$ (where $p_i(y_i|x_i)$ is a $BSC(\delta_i)$ as shown in figure). Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$.



In this problem, we show that

$$\max_{P_{\mathbf{X}}} I(\mathbf{X};\mathbf{Y}) = \sum_{i=1}^{n} (1 - h(\delta_i))$$

   (a) Show that $I(\mathbf{X};\mathbf{Y}) \le \sum_{i=1}^{n}(1 - h_2(\delta_i))$ for any $P_{\mathbf{X}}$.
   Hint: Use a chain of inequalities similar to the channel coding converse proof.

   (b) Find a distribution over $\mathbf{X}$ for which $I(\mathbf{X};\mathbf{Y}) = \sum_{i=1}^{n}(1 - h_2(\delta_i))$.

3. **Suboptimal codes**.
Consider the Z channel, described by the probability transition matrix

$$p(y|x) = \begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}.$$

Assume that we choose a $(2^{nR}, n)$ code at random, where each codeword is a sequence of *fair* coin tosses. Find the maximum rate R such that the probability of error $P_e^{(n)}$, averaged over the randomly generated codes, tends to zero as the block length n tends to infinity.

4. **Fano's inequality.** Let $\Pr(X = i) = p_i,\ i = 1, 2, \ldots, m$ and let $p_1 \geq p_2 \geq p_3 \geq \cdots \geq p_m$. The minimal probability of error predictor of X is $\hat{X} = 1$, with resulting probability of error $P_e = 1 - p_1$. Maximize $H(X)$ subject to the constraint $1 - p_1 = P_e$ to find a bound on P_e in terms of H. This is Fano's inequality in the absence of conditioning.

   Hint: Consider PMF $(p_2/P_e, p_3/P_e, \ldots, p_m/P_e)$.

5. **Modulating Switch**
   Consider the following (memoryless) channel. It has a side switch U that can be in positions ON and OFF. If U is on then the channel from X to Y is BSC_δ and if U is off then Y is Bern(1/2) regardless of X. The receiving party sees Y but not U. A design constraint is that U should be in the ON position no more than the fraction s of all channel uses, $0 \leq s \leq 1$.

   (a) One strategy is to put U into ON over the first sn time units and ignore the rest of the $(1 - s)n$ readings of Y. What is the maximal rate in bits per channel use achievable with this strategy?

   (b) Can we increase the communication rate if the encoder is allowed to modulate the U switch together with the input X (while still satisfying the s-constraint on U)? (Hint 1: Consider the problem of communication across a channel from (U^n, X^n) to Y^n.) (Hint 2: Although the channel given in Hint 1 is not iid, Fano's inequality still applies and so the converse of the channel coding theorem still holds.)

   (c) Now assume nobody has access to U, which is iid Bern(s) independent of X. Find the capacity.

6. **BSC with feedback.** Suppose that feedback is used for a binary symmetric channel with crossover probability parameter p. Each time a channel output is received, it becomes the next transmission: X_1 is Bern(1/2), $X_2 = Y_1,\ X_3 = Y_2,\ \ldots,\ X_n = Y_{n-1}$. Find $\lim_{n\to\infty} \frac{1}{n} I(X^n; Y^n)$. How does it compare to the capacity of this channel?