EE276: Homework #7

Due on Friday March 6, 6pm - Gradescope entry code: E6VP4X

1. **Method of Types and Constrained Sets.**
   Consider a finite alphabet $\mathcal{X}$. Given $D \geq 0$ and a weight function $\rho : \mathcal{X} \to \mathbb{R}_+$, define

$$B_n(\rho, D) := \left\{ x^n : \frac{1}{n} \sum_{i=1}^{n} \rho(x_i) \leq D \right\}.$$

   (a) Show that

$$B_n(\rho, D) = \bigcup_{p \in \mathbb{P}_n : \langle p, \rho \rangle \leq D} T(p)$$

   where

$$\langle p, \rho \rangle := \sum_{x \in \mathcal{X}} p(x) \rho(x).$$

   (b) Show that

$$|B_n(\rho, D)| \doteq 2^{n \max_{p : \langle p, \rho \rangle \leq D} H(p)},$$

   where $\doteq$ denotes equality up to first order in the exponent, i.e., $\alpha_n \doteq \beta_n$ means that $\lim_{n \to \infty} \frac{1}{n} \log \frac{\alpha_n}{\beta_n} = 0$. (Use the expression derived in (a) to obtain lower and upper bounds on $|B_n(\rho, D)|$ that match up to first order in exponent.)

   (c) Specializing the result of (b), show that for $D \in [0, 1/2]$,

$$\left| \left\{ y^n \in \{0,1\}^n : \frac{1}{n} \sum_{i=1}^{n} y_i \leq D \right\} \right| \doteq 2^{n h_2(D)}$$

   where h_2 is the binary entropy function.

2. **Counting.**
   Let $\mathcal{X} = \{1, 2, \ldots, m\}$. Show that the number of sequences $x^n \in \mathcal{X}^n$ satisfying

$$\frac{1}{n} \sum_{i=1}^{n} g(x_i) \geq \alpha$$

   is approximately equal to 2^{nH^*}, to first order in the exponent, for n sufficiently large, where

$$H^* = \max_{P : \sum_{i=1}^{m} P(i) g(i) \geq \alpha} H(P).$$

3. **Convexity of rate distortion function.**
   Assume $(X, Y) \sim p(x, y) = p(x) p(y|x)$. In this problem, you will show that for fixed $p(x)$, $I(X; Y)$ is a convex function of $p(y|x)$.

(a) The log sum inequality states that for n positive numbers $a_1, a_2, \cdots, a_n$, and $b_1, b_2, \cdots, b_n$, we have

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left(\sum_{i=1}^{n} a_i\right) \log \left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}\right)$$

with equality if and only if $\frac{a_i}{b_i}$ =constant. Using this inequality (you don't have to prove this inequality), show that $D(p||q)$ is convex in (p, q), i.e.,

$$\lambda D(p_1||q_1) + (1-\lambda)D(p_2||q_2) \geq D(\lambda p_1 + (1-\lambda)p_2||\lambda q_1 + (1-\lambda)q_2)$$

(b) Let $p_1(y|x)$ and $p_2(y|x)$ be two different conditional distributions. For $i \in \{1, 2\}$, let $p_i(x, y) = p_i(y|x)p(x)$, i.e., their corresponding joint distributions. For $0 \leq \lambda \leq 1$, let $p_\lambda(y|x) \triangleq \lambda p_1(y|x) + (1-\lambda)p_2(y|x)$. Show that

$$p_\lambda(y) = \lambda p_1(y) + (1-\lambda)p_2(y)$$

(c) The mutual information between random variables X and Y can be alternatively written as

$$I(X;Y) = D(p(x,y)||p(x)p(y))$$

Using this in addition to the results of the previous parts show that for fixed $p(x)$, $I(X;Y)$ is convex in $p(y|x)$.

(d) Using the previous part, show that the rate distortion function $R^{(I)}(D)$ is convex in the distortion parameter D.

4. **Properties of** $R(D)$. Consider a discrete source $X \in \mathcal{X} = \{1, 2, \ldots, m\}$ with distribution $p_1, p_2, \ldots, p_m$ and a distortion measure $d(i, j)$. Let $R(D)$ be the rate distortion function for this source and distortion measure. Let $d'(i, j) = d(i, j) - w_i$ be a new distortion measure and let $R'(D)$ be the corresponding rate distortion function. Show that $R'(D) = R(D + \bar{w})$, where $\bar{w} = \sum p_i w_i$, and use this to show that there is no essential loss of generality in assuming that $\min_{\hat{x}} d(i, \hat{x}) = 0$, i.e., for each $x \in \mathcal{X}$, there is one symbol $\hat{x}$ which reproduces the source with zero distortion.