1. A new game. You have two quarters and a table with a row of squares marked like this:

<p>| | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
<td>9</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Before the game begins, you get to place each quarter on one square. You can put either both quarters on the same square, or you can put them on two different squares: your choice. Then, you roll two fair dice, sum up the numbers showing on the dice to get a number from 2 to 12, and if there is a quarter on the square labeled with that number, remove it from the table. (If there are two quarters on that square, remove only one of them.) Now roll the two fair dice a second time, again getting a number from 2 to 12, and again removing a single quarter from the square with that number, if there is a quarter there. At this point, the game is over. If you removed both quarters, you win; if any quarter remains on the table, you lose.

a. What is the probability of winning, if you put two quarters on the square labeled 5?

b. What is your best strategy? In other words, what is the best place to put your two quarters, if you want to maximize the probability of winning? State where you should put your two quarters. Then, calculate the probability that you win, if you put your two quarters there.

2. Independence. Show that events A and B are independent if and only if $P(A|B) = P(A|B^c)$ (You need to prove that this condition is necessary and sufficient, i.e., that if independence holds, then the condition holds, and vice versa.)

3. Exponential random variable. Let $X \sim \text{Exp}(0.1)$ be the distribution of the time it takes to serve one customer at a bank (in minutes).

a. The teller has just started to serve the person ahead of you. What is the probability that you have to wait 10 minutes or more?

b. The person ahead of you has now been served for 10 minutes. What is the probability that you have to wait another 10 minutes or more?

c. For a general exponential r.v. $X \sim \text{Exp}(\lambda)$, compute $P\{X > a + b \mid X > a\}$ for $a, b \geq 0$ and interpret the result.

4. Distance to nearest star. Let the random variable $N$ be the number of stars in a region of space of volume $V$. Assume that $N$ is a Poisson random variable with pmf $p_N(n) = \frac{e^{-\rho V} (\rho V)^n}{n!}$, $n = 0, 1, 2, ...$, where $\rho$ is the “density” of stars in space. We choose an arbitrary point in space and define the random variable $X$ to be the distance from the chosen point to the nearest star. Find the pdf of $X$ in terms of $\rho$. 

5. *Generation of random variables.* You can use your favorite language to do this computational problem. We have a starter code in MATLAB for you.

(a) Given $X \sim U[0, 1]$, find the function $g(x)$ such that $Y = g(X) \sim N(2, 4)$. Plot $g(x)$.

(b) Generate 100 samples of $Y$ by first generating 100 samples of $X$ and then using the function $g(x)$ to find the corresponding $Y$ samples. Find the empirical cdf, that is, the weighted cumulative sum of the histogram of the samples of $Y$.

(c) Compare the empirical cdf to the “true” cdf of $Y$ by plotting them on the same graph.

(d) Now generate 5000 samples of $Y$ and find the empirical cdf and plot it together with the curves in part (c).

Submit the code you used and the plots.

6. *Random phase signal.* Let $Y(t) = \sin(\omega t + \Theta)$ be a sinusoidal signal with random phase $\Theta \sim U[-\pi, \pi]$. Find the pdf of the random variable $Y(t)$ for fixed values of time $t$ and radial frequency $\omega$. Comment on the dependence of the pdf of $Y(t)$ on time $t$.

7. *Binary erasure channel.* The communication channel shown in the figure below has binary input $X \sim \text{Bern}(1/3)$ and ternary output $Y \in \{0, 1, 2\}$.

```
0.9 0
0.1
0.2
0.8
1
```

```
X

0 1

2 Y
```

The conditional pmf $p_{Y|X}(y|x)$ of $Y$ given $X$ is given in the figure; e.g., $p_{Y|X}(2|0) = 0.1$.

(a) Find $p(x,y)$, $p(y)$, and $p(x|y)$ for $y = 0, 1, 2$.

(b) A decoder $\hat{\Theta}(y) \in \{0, 1\}$ decides whether $X$ is 0 or 1 from the observed output $Y$. Specify the $\hat{\Theta}(y)$ that minimizes the probability of error $P_e = P\{\hat{\Theta}(Y) \neq X\}$.

(c) Find the minimum error probability $P_e$ which is achieved by the optimal decoder.