1. **Gauss-Markov process.** Let $X_0 = 0$ and $X_n = \frac{2}{3}X_{n-1} + Z_n$ for $n \geq 1$, where $Z_1, Z_2, \ldots$ are i.i.d. $\mathcal{N}(0, 1)$. Find the mean and autocorrelation function of $X_n$.

2. **Sawtooth process.** Let $X(t) = g(t - T)$, where $g(t)$ is the periodic triangular waveform shown in Figure 1, and the delay $T$ is a random variable with $T \sim \text{U}[0, 1]$.

   ![Figure 1: Periodic triangular waveform](image)

   Is $X(t)$ a strict-sense stationary random process? Justify your answer.

3. **QAM random process.** Consider the random process

   $$X(t) = Z_1 \cos \omega t + Z_2 \sin \omega t, \quad -\infty < t < \infty,$$

   where $Z_1$ and $Z_2$ are i.i.d. discrete random variables such that $p_{Z_i}(+1) = p_{Z_i}(-1) = \frac{1}{2}$.

   a. Is $X(t)$ wide-sense stationary? Justify your answer.

   b. Is $X(t)$ strict-sense stationary? Justify your answer.

4. **Stationary Gauss-Markov process.** Consider the following variation on the Gauss-Markov process:

   $$X_0 \sim \mathcal{N}(0, a)$$
   $$X_n = \frac{1}{2}X_{n-1} + Z_n, \quad n \geq 1,$$

   where $Z_1, Z_2, Z_3, \ldots$ are i.i.d. $\mathcal{N}(0, 1)$, independent of $X_0$.

   a. Find the mean and autocorrelation functions of $X_n$.

   b. Find $a$ such that $X_n$ is stationary.
c. Consider the sample mean \( S_n = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad n \geq 1 \). Show that \( S_n \) converges to the process mean in probability even though the sequence \( X_n \) is neither i.i.d. nor uncorrelated. (Hence, \( X_n \) is a mean ergodic process.)

5. *Generating a random process with a prescribed psd.* The power spectral density \( S_X(f) \) of every WSS process is real, even, and nonnegative. In this problem you will show that, conversely, if \( S(f) \) is a real, even, nonnegative function such that \( \int_{-\infty}^{\infty} S(f)df < \infty \), then \( S(f) \) is the psd for some WSS random process. Let us consider the case that \( \int_{-\infty}^{\infty} S(f)df = 1 \).

Define the random process
\[
X(t) = \cos(2\pi Ft + \Theta),
\]
where \( F \sim S(f) \) and \( \Theta \sim U[0, 2\pi] \) are independent.

a. Show that \( X(t) \) is WSS.
b. Find the power spectral density of \( X(t) \). Interpret the result.
c. Consider the power spectral density
\[
S(f) = \frac{\alpha}{\alpha^2 + (\pi f)^2}, \quad -\infty < f < \infty.
\]
Use MATLAB to generate sample functions of \( X(t) \) for \( \alpha = 1, 5, 20 \).

6. *Windowed Poisson process.* Let \( N(t) \), for \( t \geq 0 \), be a Poisson process with rate \( \lambda > 0 \), and let \( X(t) \), for \( t \geq 0 \), be defined by \( X(t) = N(t + 1) - N(t) \). Thus, \( X(t) \) is the number of events of \( N \) during the time window \( (t, t + 1) \).

a. Sketch a typical sample path of \( N \), and the corresponding sample path of \( X \).
b. Find the mean function \( \mu_X(t) \), for \( t \geq 0 \) and the autocorrelation function \( R_X(t_1, t_2) \) for \( t_1, t_2 \geq 0 \). Express your answer in a simple form.
c. Is \( X \) a Markov process? Why or why not?
d. Consider the process \( Y(t) = (1/t) \int_0^t X(s)ds \). Determine whether \( Y(t) \) converges in the mean sense as \( t \to \infty \).

7. *Modified telegraph process.* Let \( X(t), Y(t), \) and \( W(t) \) be independent random processes; \( X(t) \) and \( Y(t) \) are zero-mean stationary Gaussian processes with \( R_X(\tau) = R_Y(\tau) = e^{-|\tau|} \). \( W(t) \) is the random telegraph process,
\[
W(t) = A(-1)^{N(t)},
\]
where \( N(t) \) is a Poisson process with parameter \( \lambda \), and the random variable

\[
A = \begin{cases} 
1 & \text{with probability 0.5} \\
-1 & \text{with probability 0.5}.
\end{cases}
\]

\( A \) and \( N(t) \) are independent. Now define the new process \( Z(t) \) as

\[
Z(t) = \begin{cases} 
X(t) & \text{if } W(t) = 1 \\
Y(t) & \text{if } W(t) = -1.
\end{cases}
\]

a. Find the first order distribution of \( Z(t) \).

b. Is \( Z(t) \) a Gaussian random process? Justify your answer.

c. Is \( Z(t) \) WSS? Justify your answer.