Homework #2
Due Thursday, April 19 at 11:59 PM

Please submit your assignment as a PDF to the class Gradescope page.

1. **Schwarz Inequality** We will use the Schwarz Inequality to show that $|\rho_{X,Y}| = 1$ if and only if $(X - E(X))$ is a linear function of $(Y - E(Y))$, where $\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$, and the triangle inequality.
   a. Prove the following inequality, which is known as the *Schwarz inequality*.
   $$(E(XY))^2 \leq E(X^2)E(Y^2).$$
   Hint: Use the fact that $E((X + aY)^2) \geq 0$ for every real number $a$.
   b. Prove that equality holds if and only if either $Y = cX$ or $X = cY$ for some constant $c$.
   c. Use the Schwarz inequality to show that the correlation coefficient $\rho_{x,y}$ satisfies $|\rho_{x,y}| \leq 1$.
   d. Show that $|\rho_{X,Y}| = 1$ if and only if $(X - E(X))$ is a linear function of $(Y - E(Y))$ i.e. $(X - E(X)) = c(Y - E(Y))$ or $(Y - E(Y)) = c(X - E(X))$.
   e. Show that
   $$E((X + Y)^2) \leq (2\sqrt{E(X^2)} + 2\sqrt{E(Y^2)})^2.$$ 
   This is called the *triangle inequality*.

2. **Radar signal detection.** The received signal $S$ for a radar channel is 0 if there is no target and a random variable $X \sim \mathcal{N}(0, P)$ if there is a target. Both possibilities occur with equal probability. Thus
   $$S = \begin{cases} 0 & \text{with probability } \frac{1}{2} \\ X \sim \mathcal{N}(0, P) & \text{with probability } \frac{1}{2}. \end{cases}$$
   The radar receiver observes $Y = S + Z$, where the noise $Z \sim \mathcal{N}(0, N)$ is independent of $S$. Find the optimal decoder for deciding whether $S = 0$ or $S = X$ and its probability of error. Give your answer in terms of intervals of $y$ and express the boundary points of the intervals in terms of $P$ and $N$.
   Hint: You can cast this detection problem in the form discussed in class by defining the signal $\Theta$ to be 0 if $S = 0$ and 1 if $S = X$.

3. **Ternary signaling.** Let the signal $S$ be a random variable defined as follows:
   $$S = \begin{cases} -1 & \text{with probability } \frac{1}{3} \\ 0 & \text{with probability } \frac{1}{3} \\ +1 & \text{with probability } \frac{1}{3}. \end{cases}$$
   The signal is sent over a channel with additive Laplacian noise $Z$, i.e., $Z$ is a Laplacian random variable with pdf
   $$f_Z(z) = \frac{\lambda}{2} e^{-\lambda|z|}, \quad -\infty < z < \infty.$$
The signal $S$ and the noise $Z$ are assumed to be independent and the channel output is their sum $Y = S + Z$.

a. Find $f_{Y|S}(y|s)$ for $s = -1, 0, +1$. Sketch the conditional pdfs on the same graph.

b. Find the optimal decoding rule $\hat{Θ}(Y)$ for deciding whether $S$ is $-1, 0$ or $+1$. Give your answer in terms of ranges of values of $Y$.

c. Find the probability of decoding error for $\hat{Θ}(y)$ in terms of $\lambda$.

4. Function of uniform random variables. Let $X$ and $Y$ be two independent $U[0, 1]$ random variables. Find the probability density function (pdf) of $Z = (X + Y) \mod 1$ (i.e., $Z = X + Y$ if $X + Y \leq 1$ and $X + Y - 1$ if $X + Y > 1$).

5. Jensen’s inequality. A function $g(x)$ is said to be convex on an interval $(a, b)$ if for every $x_1, x_2$ in $(a, b)$ and for every $λ$ satisfying $0 \leq λ \leq 1$,

$$g(λx_1 + (1 - λ)x_2) \leq λg(x_1) + (1 - λ)g(x_2).$$

Further, $g(x)$ is said to be strictly convex if equality holds only for $λ = 0$ and $λ = 1$. It can be shown that if $g(x)$ is twice differentiable, then it is convex iff $g''(x) ≥ 0$ for all $x$ in $(a, b)$ and strictly convex iff $g(x) > 0$ for all $x$ in $(a, b)$. See Figure 3.

a. Show that if $g(x)$ is convex on $(a, b)$ and $X ∈ X \subset (a, b)$ is a discrete random variable, then

$$E(g(X)) ≥ g(E(X)).$$

Hint: Use induction on the number of $X$ values with nonzero probability.

For the following parts, find the inequality relationship ($≤$ or $≥$) and justify your answer.

b. $E(e^{2X})$ and $e^{E(2X)}$.

c. $E(\ln X)$ and $\ln(E(X))$ for $X ≥ 0$

d. $(E(X^2))^6$ and $E(X^{12})$.

Figure 1: Convex function illustrating $g(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda g(x_1) + (1 - \lambda)g(x_2)$.
The following problems are optional and need not be turned in for grading.

1. **Mean-square inequality.** Let $X$ and $Y$ be random variables with finite means and variances. Show that

$$ P\{|X - Y| > \epsilon\} \leq \frac{E((X - Y)^2)}{\epsilon^2}. $$

2. **Joint cdf or not.** Consider the function

$$ G(x, y) = \begin{cases} 
1 & \text{if } x + y \geq 0 \\
0 & \text{otherwise.}
\end{cases} $$

Can $G$ be a joint cdf for a pair of random variables? Justify your answer.

3. **Max to Min ratio.** Let $X_1$ and $X_2$ be two independent random variables, each uniformly distributed between 0 and 1, i.e., $X_i \sim U[0, 1]$. Find and sketch the cdf of

$$ Y = \max(X_1, X_2) / \min(X_1, X_2). $$

4. **Mean square error estimation.** The number of packets arriving per unit time at a node in a communication network is a Poisson random variable $X$ with rate $\Lambda \sim \text{Exp}(a)$. Find the MMSE estimate of the rate $\Lambda$ given the observation $X$. Your answer should be in terms only of $X$ and the constant $a$. Hint: you do not need to evaluate complicated integrals here. Just use integration by parts, i.e., $\int_\alpha^\beta udv = uv|_\alpha^\beta - \int_\alpha^\beta vdu$. The final result will look very nice!