Homework #3
Due Thursday, April 26 at 11:59 PM

Please submit your assignment as a PDF to the class Gradescope page.

1. **Shot noise channel.** Consider an additive noise channel with input signal \( X \sim U(0, 1) \) and output signal \( Y = X + Z \), where the noise \( Z|X = x \sim N(0, ax) \), for some constant \( a > 0 \), i.e., the noise variance is proportional to the signal. Observing \( Y \), find the MMSE linear estimate of \( X \). Your answer should be in terms of only \( a \) and \( Y \).

2. **Additive-noise channel with path gain.** Consider the output \( Y \) of an additive-noise channel with path gain, where \( X \) and \( Z \) are zero mean and uncorrelated, and \( a \) and \( b \) are constants. Find the MMSE linear estimate of \( X \) given \( Y \) and its MSE in terms only of \( \sigma_X^2, \sigma_Z^2, a, \) and \( b \).

![Figure 1: Channel for problem 2](image)

3. **Camera measurement.** The measurement from a camera can be expressed as \( Y = AX + Z \), where \( X \) is the object position with mean \( \mu \) and variance \( \sigma_X^2 \), \( A \) is the occlusion indicator function and is equal to 1 (if the camera can see the object) with probability \( p \), and 0 (if the camera cannot see the object) with probability \( (1 - p) \), and \( Z \) is the measurement error with mean 0 and variance \( \sigma_Z^2 \). Assume that \( X, A, \) and \( Z \) are independent. Find the best linear MSE estimate of \( X \) given the camera measurement \( Y \). Your answer should be in terms of only \( \mu, \sigma_X^2, \sigma_Z^2, \) and \( p \).

4. **Jointly Gaussian random variables.** Let \( X \) and \( Y \) be jointly Gaussian random variables with mean 0 and covariance matrix

\[
\begin{bmatrix}
\sigma_X^2 & \sigma_X \sigma_Y \rho_{X,Y} \\
\sigma_X \sigma_Y \rho_{X,Y} & \sigma_Y^2
\end{bmatrix}
\]

a. What is the pdf of \( E(X | Y) \)?

b. What is the minimum MSE estimate of \( Y^2 \) given \( X \)?

Your answers should be in terms of \( \sigma_X, \sigma_Y, \rho_{X,Y}, \) and the random variables \( X \) and \( Y \).

5. **Estimation vs. detection.** Signal \( X \) and noise \( Z \) are independent random variables, where

\[ X = \begin{cases} 
+1 & \text{with probability } \frac{1}{2} \\
-1 & \text{with probability } \frac{1}{2}, 
\end{cases} \]

and \( Z \sim U[-2, +2] \). Their sum \( Y = X + Z \) is observed.

a. Find the minimum MSE estimate of \( X \) given \( Y \) and the corresponding mean square error. What is the probability of error of this estimate?
b. Suppose that we decide whether $X = +1$ or $X = -1$ using a decoder that minimizes the probability of error. Find this optimal decoder and its probability of error. Compare the optimal decoder’s MSE to the minimum MSE.

6. *Jointly Gaussian random variables, redux.* Consider the following joint pdf for $X$ and $Y$:

$$f_{X,Y}(x, y) = \frac{1}{\pi \sqrt{3/4}} e^{-\frac{1}{4} \left( \frac{1}{4} x^2 + \frac{1}{4} y^2 + \frac{1}{2} xy - 8x - 16y + 16 \right)}$$

a. Find $E(X)$, $E(Y)$, $\text{Var}(X)$, $\text{Var}(Y)$, and $\text{Cov}(X,Y)$.

b. Find the minimum MSE estimate of $X$ given $Y$ and the corresponding MSE.

7. *Conditional Independence does not imply Independence.* In class, we saw an example in which two independent, identically distributed random variables conditioned on a third random variable were no longer independent. Here, we examine an example of the opposite case: is it possible for conditionally independent random variables to be not independent?

Suppose $X_3 \sim \text{U}[0, 1]$, given $X_3$, $X_1$, $X_2$ i.i.d. $\text{Bern}(X_3)$. Show that $X_1$, $X_2$ are not independent.

Work out the joint distribution $P_{X_1, X_2}$.

Hint: the Beta function is $B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$.

The following problems are optional and need not be turned in for grading.

1. *Independence vs. Conditional Independence* Give an example of random variables $X, Y,$ and $Z$ where $f_{X,Z}(x, z) = f_X(x)f_Z(z)$ but $f_{X,Z|Y}(x, z|y) \neq f_X|Y(x|y)f_Z|Y(z|y)$ i.e. independence does not imply conditional independence.

2. *Sum and difference.* Let $X$ and $Y$ be two random variables, and define $U = X - Y$ and $V = X + Y$. Find the minimum MSE linear estimate of $V$ given $U$ as a function of the random variables and $E(X)$, $E(Y)$, $\sigma_X$, $\sigma_Y$, $\rho_{X,Y}$, where $\sigma_X = \sqrt{\text{Var}(X)}, \rho_{X,Y} = \text{corr}(X,Y)$.

3. *Covariance matrices.* Which of the following matrices can be a covariance matrix? Justify your answer. Either construct a random vector $\mathbf{X}$ with the given covariance matrix as a function of the i.i.d. zero mean unit variance random variables $Z_1, Z_2, Z_3$, or establish a contradiction as was done in lecture.

$$
\begin{align*}
\text{(a)} & \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} & \text{(b)} & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \text{(c)} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} & \text{(d)} & \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}
\end{align*}
$$