Homework #5
Due Thursday, May 17 at 11:59 PM

Please submit your assignment as a PDF to the class Gradescope page.

1. An innovations sequence and its applications. Let $\begin{bmatrix} Y_1 & Y_2 & Y_3 & X \end{bmatrix}^\top$ be a zero-mean random vector with covariance matrix

$$
\begin{bmatrix}
1 & 0.5 & 0.5 & 0 \\
0.5 & 1 & 0.5 & 0.25 \\
0.5 & 0.5 & 1 & 0.25 \\
0 & 0.25 & 0.25 & 1
\end{bmatrix}.
$$

a. Let $\tilde{\mathbf{Y}} = [\tilde{Y}_1 \  \tilde{Y}_2 \  \tilde{Y}_3]^\top$ be the innovations sequence of $\tilde{\mathbf{Y}} = [Y_1 \  Y_2 \  Y_3]^\top$. Find the matrix $A$ such that

$$
\tilde{\mathbf{Y}} = A \mathbf{Y}.
$$

b. Find the covariance matrix of $\tilde{\mathbf{Y}}$ and the cross-covariance matrix of $X$ and $\tilde{\mathbf{Y}}$.

c. Find the constants $a, b$, and $c$ that minimize $E[(X - a\tilde{Y}_1 - b\tilde{Y}_2 - c\tilde{Y}_3)^2]$.

2. Cellphone. We aim to design a cellphone which is able to denoise a signal modeled as $Y_1 = V + Z$, where $V$ is a random variable representing the user’s voice and $Z$ is a random variable representing background noise. An extra microphone measures the background. However this measurement also includes some distorted voice signal. This is taken into account by modeling it as $Y_2 = Z + U$, where $U$ is a random variable representing the distortion. Assume that $V$, $Z$ and $U$ are all zero mean, both $(V, Z)$ and $(U, Z)$ are uncorrelated and $\text{Corr}(U, V) = \rho$. We also know that $\text{Var}(V) = P$, $\text{Var}(Z) = N$ and $\text{Var}(U) = Q$.

We decide to obtain a linear estimate of $V$ from $Y_1$ and $Y_2$.

![Figure 1: Illustration of the system](image)
a. What is the innovation sequence $\tilde{Y}_1$ and $\tilde{Y}_2$ corresponding to $Y_1$ and $Y_2$?

b. What is the linear MMSE estimate of $V$ given the measurements expressed as a function of $\tilde{Y}_1$, $\tilde{Y}_2$, $P$, $Q$ and $N$?

c. What is the corresponding MSE in terms of $P$, $Q$ and $N$?

3. **Vector Kalman filter experiment.** Consider the state space model

$$X_{i+1} = \begin{bmatrix} 1 & 1 \\ 0 & 0.9 \end{bmatrix} X_i + U_i, \quad \text{for} \quad i = 1, \ldots, n.$$  

The first and second component of $X_i$ are the one-dimensional position and velocity of a moving object. In each step, the position and velocity evolve according to Newton’s laws of physics. Due to friction, the velocity is dampened by a factor of 0.9 in each time step. The state is disturbed by independent noise vectors $U_i$, distributed according to

$$U_i \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix} \right), \quad \text{for} \quad i = 1, \ldots, n.$$  

The initial state is independent of the noise vectors $U_i$ and distributed according to

$$X_1 \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1000 & 0 \\ 0 & 0 \end{bmatrix} \right)$$  

The observations are

$$Y_i = X_i + V_i, \quad \text{for} \quad i = 1, \ldots, n + 1,$$

where the noise $V_i$ is distributed according to

$$V_i \sim \mathcal{N} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1000 & 0 \\ 0 & 1 \end{bmatrix} \right), \quad \text{for} \quad i = 1, \ldots, n + 1,$$

independent of the initial state $X_1$ and the state noise vectors $U_i$.

Download the posted MATLAB file *vectorKalman.m* and complete the code to compute the Kalman prediction filter $\hat{X}_{i+1\mid i}$ from the observations $Y^i$, for $i = 0, \ldots, n+1$. For a single realization with time horizon $n = 100$, plot the true state and its prediction over the time index $i$. In a separate figure, plot the prediction error over $i$. Hand in your MATLAB code and the plots.
4. **The filtering version of the Kalman filter** We have studied the derivation of scalar Kalman filter to predict the next state in class. Now we are interested in estimating the current state. Derive the update equations and the error:

\[
\hat{X}_{i+1|i+1} = a_i(1 - k_i)\hat{X}_{i|i} + k_i Y_{i+1}
\]

where

\[
k_i = \frac{a_i^2 \sigma_{i|i}^2 + Q_i}{a_i^2 \sigma_{i|i}^2 + Q_i + N_{i+1}}
\]

and

\[
\sigma_{i+1|i+1} = (1 - k_i)(a_i^2 \sigma_{i|i}^2 + Q_i).
\]

5. **Absolute value random walk.** Let \(X_n\) be a random walk defined by \(X_0 = 0, X_n = \sum_{i=1}^{n} Z_i, n \geq 1\), where \(\{Z_i\}\) is an i.i.d. process with \(P\{Z_1 = -1\} = P\{Z_1 = +1\} = \frac{1}{2}\).

Define the absolute value random process \(Y_n = |X_n|\).

a. Find \(P\{Y_n = k\}\).

b. Find \(P\{\max\{Y_i : 1 \leq i < 20\} = 10 | Y_{20} = 0\}\).

6. **Poisson process branching.** Let \(N(t)\) be a Poisson process with rate \(\lambda\). We split \(N(t)\) into two counting subprocesses \(N_1(t)\) and \(N_2(t)\) such that \(N(t) = N_1(t) + N_2(t)\) as follows: each event is randomly and independently assigned to process \(N_1(t)\) with probability \(p\), otherwise it is assigned to \(N_2(t)\). Prove that \(N_1(t)\) is a Poisson process with rate \(p\lambda\) and \(N_2(t)\) is a Poisson process with rate \((1 - p)\lambda\).

7. **Markov processes.** Let \(\{X_n\}\) be a discrete-time continuous-valued Markov random process, that is,

\[f(x_{n+1}|x_1, x_2, \ldots, x_n) = f(x_{n+1}|x_n)\]

for every \(n \geq 1\) and for all sequences \((x_1, x_2, \ldots, x_{n+1})\).

a. Show that \(f(x_1, \ldots, x_n) = f(x_1)f(x_2|x_1) \cdots f(x_n|x_{n-1}) = f(x_n)f(x_{n-1}|x_n) \cdots f(x_1|x_2)\).

b. Show that \(f(x_n|x_1, x_2, \ldots, x_k) = f(x_n|x_k)\) for every \(k\) such that \(1 \leq k < n\).

c. Show that \(f(x_{n+1}, x_{n-1}|x_n) = f(x_{n+1}|x_n)f(x_{n-1}|x_n)\), that is, the past and the future are independent given the present.