1. Applications of cross-correlation. Estimating the cross-correlation between two WSS random processes has several real-world signal processing applications. In this problem we illustrate two such applications.

a. Finding the impulse response of an LTI system. To find the impulse response $h(t)$ of an LTI system (e.g. a concert hall), i.e. to identify the system, white noise $X(t)$, $-\infty < t < \infty$, is applied to its input and the output $Y(t)$ is measured. Given the input and the output sample functions, the cross-correlation $R_{YX}(\tau)$ is estimated. Show how $R_{YX}(\tau)$ can be used to find $h(t)$.

b. Finding time of flight. Finding the distance to an object is often done by sending a signal and measuring the time of flight, i.e. the time it takes for the signal to return (assuming speed of signal, e.g. light, is known). Let $X(t)$ be the signal sent and $Y(t) = X(t-\delta) + Z(t)$ be the signal received, where $\delta$ is the unknown time of flight. Assume that $X(t)$ and $Z(t)$ (the sensor noise) are uncorrelated zero mean WSS processes. The estimated cross-correlation function of $Y(t)$ and $X(t)$, $R_{YX}(t)$ is shown in Figure 1. Find the time of flight $\delta$.

Figure 1: Cross-correlation function for problem 1b

Solution (10 points)

a. Since white noise has a flat PSD, the crosspower spectral density of the input $X(t)$ and the output $Y(t)$ is just the transfer function of the system scaled by the PSD of the white noise.

\[ S_{YX}(f) = H(f)S_X(f) = H(f) \frac{N_0}{2} \]

\[ R_{YX}(\tau) = \mathcal{F}^{-1}(S_{YX}(f)) = \frac{N_0}{2} h(\tau). \]
Thus to estimate the impulse response of a linear time-invariant system, we apply white noise to its input, estimate the cross-correlation function of its input and output, and scale by $2/N_0$.

b. The cross-correlation function of $Y(t)$ and $X(t)$ is
\[
R_{YX}(\tau) = E[Y(t+\tau)X(t)] \\
= E[(X(t-\delta+\tau) + Z(t+\tau))X(t)] \\
= R_X(\tau-\delta).
\]

Since the maximum of $|R_X(\alpha)|$ is achieved for $\alpha = 0$, by inspection of the given $R_{XY}$ we see that $5 - \delta = 0$. Thus $\delta = 5$.

2. **Switched RC circuit.** Consider the circuit in Figure 2. The voltage source $V(t)$ models the thermal noise in the resistor. At time $t = 0$, the switch is closed. Compute the average power $E[V_o^2(t)]$ as a function of time $t$.

Lecture Notes 8 slides 15 and 16 show the computation of the average output noise power of an RC circuit. Compare that result to your answer in this problem.

**Solution** (10 points)

Let us lump the voltage source and the switch together as a (non-stationary) source that drives the linear time-invariant RC system. The autocorrelation function of this source is
\[
R_V(t_1, t_2) = \begin{cases} 
2kT R \delta(t_1 - t_2) & \text{if } t_1, t_2 \geq 0 \\
0 & \text{otherwise.}
\end{cases}
\]

The impulse response of the LTI system composed of the resistor and the capacitor is
\[
h(t) = \frac{1}{RC} e^{-t/(RC)} u(t).
\]

The output random process is given by the convolution
\[
V_o(t) = \int_{-\infty}^{\infty} h(\tau)V(t - \tau)d\tau.
\]
We can compute the output power as a function of $t$ as

$$E[V_o^2(t)] = E \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)V(t-\tau_1)V(t-\tau_2)d\tau_2d\tau_1 \right]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_V(t-\tau_1, t-\tau_2)d\tau_2d\tau_1$$

$$= 2kTR \int_{-\infty}^{t} h(\tau_1)h(\tau_1)d\tau_1$$

$$= 2kTR \int_{-\infty}^{t} e^{-2\tau_1/(RC)}d\tau_1$$

$$= \frac{2kT}{RC^2}\left[\frac{RC}{2} - \frac{e^{-2t/(RC)}}{2}\right]_0$$

$$= \frac{kT}{C}(1 - e^{-2t/(RC)}).$$

Note that as $t \to \infty$, the average output power converges to $kT/C$.

3. **LTI system with WSS process input.** Let $Y(t) = h(t) * X(t)$ and $Z(t) = X(t) - Y(t)$, as shown in Figure 3.

a. Find $S_Z(f)$.

b. Find $E[Z^2(t)]$.

Your answers should be in terms of $S_X(f)$ and the transfer function $H(f) = \mathcal{F}\{h(t)\}$.

![Figure 3: Linear time-invariant system](image)

**Solution** (10 points)

a. To find $S_Z(f)$, we start with

$$Z(t) = (\delta(t) - h(t)) * X(t).$$

Taking the power spectral density on both sides, we arrive at

$$S_Z(f) = |1 - H(f)|^2S_X(f).$$

b. The average power of $Z(t)$ is the area under $S_Z(f)$,

$$E[Z^2(t)] = \int_{-\infty}^{\infty} |1 - H(f)|^2S_X(f)df.$$
4. **Linear system with feedback.** Given the system in Figure 4 where $X(t)$ and $Z(t)$ are independent zero-mean WSS random processes, and $h(t)$ is the impulse response of a linear time invariant system, find the power spectral density of $Y(t)$.

![Figure 4: Linear system with feedback](image)

**Solution** (10 points)

It follows from the block diagram that

$$(X(t) - Y(t)) * h(t) + Z(t) = Y(t).$$

Sorting terms, this is equivalent to

$$X(t) * h(t) + Z(t) = (\delta(t) + h(t)) * Y(t).$$

Taking the power spectral density on both sides, this implies

$$S_X(f)|H(f)|^2 + S_Z(f) = |1 + H(f)|^2 S_Y(f),$$

where we have used the fact that $X(t)$ and $Z(t)$ are independent, and the power spectral density of the sum is thus the sum of the power spectral densities. Finally, we conclude that

$$S_Y(f) = \frac{|H(f)|^2}{|1 + H(f)|^2} S_X(f) + \frac{1}{|1 + H(f)|^2} S_Z(f).$$

5. **Discrete-time LTI system with white noise input.** Let $\{X_n : -\infty < n < \infty\}$ be a discrete-time white noise process, i.e., $E[X_n] = 0$ and

$$R_X(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The process is filtered using a linear time-invariant system with impulse response

$$h(n) = \begin{cases} \alpha & n = 0 \\ \beta & n = 1 \\ 0 & \text{otherwise.} \end{cases}$$
Find $\alpha$ and $\beta$ such that the output process $Y_n$ has

\[
R_Y(n) = \begin{cases} 
2 & n = 0 \\
1 & |n| = 1 \\
0 & \text{otherwise}.
\end{cases}
\]

**Solution** (10 points)

We are given that $R_X(n)$ is a discrete-time unit impulse. Therefore

\[
R_Y(n) = h(n) * R_X(n) * h(-n) = h(n) * h(-n).
\]

The impulse response $h(n)$ is the sequence $(a, \beta, 0, 0, \ldots)$. The convolution with $h(-n)$ has only finitely many nonzero terms.

\[
\begin{align*}
R_Y(0) &= 2 = h(0) * h(0) = \alpha^2 + \beta^2 \\
R_Y(+1) &= 1 = h(1) * h(-1) = \alpha \beta \\
R_Y(-1) &= 1 = R_Y(1)
\end{align*}
\]

This pair of equations has two solutions: $\alpha = +1$ and $\beta = +1$ or $\alpha = -1$ and $\beta = -1$.

6. **Narrow-band process over additive white noise channel.** Let the received signal over an additive noise channel be $Y(t) = X(t) + Z(t)$. The input signal $X(t)$ is a WSS process with zero mean and autocorrelation function $R_X(\tau) = P \cos(10\pi \tau) \cdot \text{sinc}(\tau)$. The noise $Z(t)$ is a white noise process with power spectral density $S_Z(f) = N/2$, $-\infty < f < \infty$. The signal and noise processes are uncorrelated.

a. Find and sketch the transfer function of the best infinite smoothing filter for $X(t)$ given $Y(\tau)$, $-\infty < \tau < \infty$.

b. Find the MSE of the best infinite smoothing filter.

Your answers should be in terms of only $P$ and $N$.

**Solution** (15 points)

a. We first find $S_{XY}(f)$ and $S_Y(f)$. Since $X(t)$ and $Z(t)$ are zero mean and uncorrelated,

\[
\begin{align*}
R_{XY}(\tau) &= R_X(\tau), \\
S_{XY}(f) &= \frac{P}{2} (\text{Rect}(f - 5) + \text{Rect}(f + 5)), \\
R_Y(\tau) &= R_X(\tau) + R_Z(\tau), \\
S_Y(f) &= \frac{P}{2} (\text{Rect}(f - 5) + \text{Rect}(f + 5)) + \frac{N}{2}.
\end{align*}
\]
The transfer function of the best infinite smoothing filter is

\[ H(f) = \frac{S_{XY}(f)}{S_Y(f)} \]

\[ = \frac{P}{N + P} (\text{Rect}(f - 5) + \text{Rect}(f + 5)). \]

This is sketched in Figure 5.

![Figure 5: Transfer function \( H(f) \)](image)

b. The MSE is given by

\[ \text{MSE} = \int_{-\infty}^{\infty} \left( S_X(f) - \frac{|S_{XY}(f)|^2}{S_Y(f)} \right) df \]

\[ = 2 \int_{4.5}^{5.5} \left( \frac{P}{2} - \frac{(P/2)^2}{N/2 + P/2} \right) df \]

\[ = \frac{NP}{P + N} \]