Section 2

EE278: Introduction to Statistical Signal Processing (Fall 2020)

Monday, Sep 28, 2020 - 3 to 4 pm

1. Equivalent Definitions of a Positive Semi-definite Matrix

For an $n \times n$ symmetric matrix A, show that the following are equivalent:

(1) All eigenvalues of A are non-negative.

(2) $A = U^T U$ for some $n \times n$ matrix U.

(3) $v^T A v \geq 0$ for all vectors $v \in \mathbb{R}^n$.

Hint: Show that (1) $\implies$ (2), (2) $\implies$ (3), and (3) $\implies$ (1).

2. Square Root of a PSD Matrix

Let A be a positive semi-definite matrix. Find another positive semi-definite matrix B such that $A = B^2$.

Hint: Use the fact that $A = Q\Lambda Q^T$ for some diagonal Λ and orthonormal Q.

Note: Such a B is called the principal or non-negative square root of A (denoted as $A^{1/2}$). For a positive semi-definite A, there is a unique principal square root B.

3. Whitening/Coloring Transformations

(a) Let $\mathbf{W}$ be an i.i.d. normalized Gaussian random vector. Find a linear transformation A such that the random vector $\mathbf{Y} = A\mathbf{W}$ has the covariance matrix K.

(b) Let $\mathbf{X}$ be a zero-mean jointly Gaussian random vector with covariance matrix K. Find a linear transformation A such that the random vector $\mathbf{Y} = A\mathbf{X}$ has the covariance matrix I (identity). Assume that K is full rank.

(c) Let $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, K_1)$. Find a linear transformation A such that $\mathbf{Y} = A\mathbf{X} \sim \mathcal{N}(\mathbf{0}, K_2)$. Assume that K_1 is full rank.

4. Covariance Matrices

Which of the following matrices can be a covariance matrix? Justify your answer. Either construct a random vector $\mathbf{X}$ with the given covariance matrix as a function of the i.i.d. zero mean unit variance random variables Z_1, Z_2, Z_3, or establish a contradiction.

$$\text{(a)} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{(c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad \text{(d)} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

5. Properties of Symmetric Matrices (not a question)

If A is an $n \times n$ symmetric matrix,

1. Every eigenvalue λ of A is real and has a real eigenvector q corresponding to it, i.e. $Aq = \lambda q$ (eigenvalues may be repeated).

2. Eigenvectors corresponding to distinct eigenvalues are orthogonal, i.e. for $\lambda_1 \neq \lambda_2$,

$$Aq_1 = \lambda_1 q_1, Aq_2 = \lambda_2 q_2 \implies q_1^T q_2 = 0$$

3. Let Λ be the diagonal matrix whose diagonal entries are the eigenvalues of A. Let Q be the orthonormal matrix (i.e. $Q^T Q = QQ^T = I$) whose columns are the corresponding normalized eigenvectors (i.e. $\|q_i\| = 1$). Then

$$A = Q\Lambda Q^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

   This is called the eigenvalue decomposition or spectral decomposition of A.