Section 2

EE278: Introduction to Statistical Signal Processing (Fall 2020)

Monday, Sep 28, 2020 - 3 to 4 pm

1. Equivalent Definitions of a Positive Semi-definite Matrix

For an $n \times n$ symmetric matrix A, show that the following are equivalent:

(1) All eigenvalues of A are non-negative.

(2) $A = U^T U$ for some $n \times n$ matrix U.

(3) $v^T A v \geq 0$ for all vectors $v \in \mathbb{R}^n$.

Hint: Show that (1) $\implies$ (2), (2) $\implies$ (3), and (3) $\implies$ (1).

Solution:

To show (1) $\implies$ (2), write $A = Q\Lambda Q^T$ for a diagonal Λ and orthonormal Q, since A is symmetric. Then choose $U = \Lambda^{1/2} Q^T$, where $\Lambda^{1/2}$ is the element-wise square root of Λ. This can be done since all eigenvalues (entries of Λ) are non-negative.

To show (2) $\implies$ (3), observe that

$$v^T A v = v^T U^T U v = \|Uv\|^2 \geq 0 \quad \forall\, v \in \mathbb{R}^n$$

Finally to show that (3) $\implies$ (1), choose v as one of the eigenvectors of A with eigenvalue λ. Then

$$v^T A v = v^T(\lambda v) = \lambda v^T v = \lambda \|v\|^2 \geq 0$$

Since $\|v\|^2 \geq 0$, this means that $\lambda \geq 0$.

2. Square Root of a PSD Matrix

Let A be a positive semi-definite matrix. Find another positive semi-definite matrix B such that $A = B^2$.

Hint: Use the fact that $A = Q\Lambda Q^T$ for some diagonal Λ and orthonormal Q.

Note: Such a B is called the principal or non-negative square root of A (denoted as $A^{1/2}$). For a positive semi-definite A, there is a unique principal square root B.

Solution:

Since A is positive semi-definite, write $A = Q\Lambda Q^T$ where Q is orthonormal and λ is a diagonal matrix with non-negative entries. Then choose $B = Q\Lambda^{1/2} Q^T$ where $\Lambda^{1/2}$ is the element-wise square root of Λ. Since $Q^T Q = 1$,

$$B^2 = \left(Q\Lambda^{1/2}Q^T\right)\left(Q\Lambda^{1/2}Q^T\right) = Q\Lambda^{1/2}\Lambda^{1/2}Q^T = Q\Lambda Q^T = A$$

3. Coloring/Whitening Transformations

(a) Let $\mathbf{W}$ be an i.i.d. normalized Gaussian random vector. Find a linear transformation A such that the random vector $\mathbf{Y} = A\mathbf{W}$ has the covariance matrix K.

(b) Let $\mathbf{X}$ be a zero-mean jointly Gaussian random vector with covariance matrix K. Find a linear transformation A such that the random vector $\mathbf{Y} = A\mathbf{X}$ has the covariance matrix I (identity). Assume that K is full rank.

(c) Let $\mathbf{X} \sim \mathcal{N}(\mathbf{0}, K_1)$. Find a linear transformation A such that $\mathbf{Y} = A\mathbf{X} \sim \mathcal{N}(\mathbf{0}, K_2)$. Assume that K_1 is full rank.

Solution:

(a) Choose $A = K^{1/2} = Q\Lambda^{1/2}Q^T$ where $K = Q\Lambda Q^T$. Then covariance of $\mathbf{Y}$ is

$$K_{\mathbf{Y}} = K^{1/2}K_{\mathbf{W}}(K^{1/2})^T = K^{1/2}IK^{1/2} = K$$

If a symmetric transformation is not required, you may also choose $A = Q\Lambda^{1/2}$. Then

$$K_{\mathbf{Y}} = Q\Lambda^{1/2}I(\Lambda^{1/2})^T Q^T = Q\Lambda Q^T = K$$

Thus, this transformation is not unique. This can be called a "coloring" transformation because it changes the covariance from identity to an arbitrary one.

(b) Choose $A = K^{-1/2} = Q\Lambda^{-1/2}Q^T$ where $K = Q\Lambda Q^T$. This is well-defined when all eigenvalues are positive, i.e. K is full rank. Then covariance of $\mathbf{Y}$ is

$$\begin{aligned}
K_{\mathbf{Y}} &= K^{-1/2}KK^{-1/2} \\
&= \left(Q\Lambda^{-1/2}Q^T\right)\left(Q\Lambda Q^T\right)\left(Q\Lambda^{-1/2}Q^T\right) \\
&= Q\Lambda^{-1/2}\Lambda\Lambda^{-1/2}Q^T \\
&= QQ^T \\
&= I
\end{aligned}$$

If a symmetric transformation is not required, you may also choose $A = \Lambda^{-1/2}Q^T$. Then

$$\begin{aligned}
K_{\mathbf{Y}} &= \Lambda^{-1/2}Q^T KQ(\Lambda^{-1/2})^T \\
&= \Lambda^{-1/2}Q^T\left(Q\Lambda Q^T\right)Q\Lambda^{-1/2} \\
&= \Lambda^{-1/2}\Lambda\Lambda^{-1/2} \\
&= I
\end{aligned}$$

Thus, this transformation is also not unique. This can be called a "whitening" transformation because it changes the covariance to identity. The term white is used because just as white light has equal intensity in all frequencies, this random vector has equal variance in all directions.

(c) Choose any matrix $A = A_2A_1$ where A_1 is a whitening matrix for K_1 (part (b)) and A_2 is a coloring matrix for K_2 (part (a)).

4. Covariance Matrices

Which of the following matrices can be a covariance matrix? Justify your answer. Either construct a random vector $\mathbf{X}$ with the given covariance matrix as a function of the i.i.d. zero mean unit variance random variables Z_1, Z_2, Z_3, or establish a contradiction.

$$\text{(a)}\ \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \qquad \text{(b)}\ \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{(c)}\ \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \qquad \text{(d)}\ \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 3 \end{bmatrix}$$

Solution:

a) No: not symmetric.

b) Yes: covariance matrix of $X_1 = Z_1 + Z_2$ and $X_2 = Z_1 + Z_3$.

c) Yes: covariance matrix of $X_1 = Z_1,\ X_2 = Z_1 + Z_2,$ and $X_3 = Z_1 + Z_2 + Z_3$.

d) No: several justifications.

- $\sigma_{23}^2 = 9 > \sigma_{22}\sigma_{33} = 6$, which contradicts the Schwarz inequality i.e. $\text{Cov}(X_2, X_3)^2 \le \text{Var}(X_2)\text{Var}(X_3)$.
- The matrix is not positive semi-definite since the determinant is -2.
- One of the eigenvalues is negative ($\lambda_1 = -0.8056$).

5. Properties of Symmetric Matrices (not a question)

If A is an $n \times n$ symmetric matrix,

1. Every eigenvalue λ of A is real and has a real eigenvector q corresponding to it, i.e. $Aq = \lambda q$ (eigenvalues may be repeated).

2. Eigenvectors corresponding to distinct eigenvalues are orthogonal, i.e. for $\lambda_1 \neq \lambda_2$,

$$Aq_1 = \lambda_1 q_1, Aq_2 = \lambda_2 q_2 \implies q_1^T q_2 = 0$$

3. Let Λ be the diagonal matrix whose diagonal entries are the eigenvalues of A. Let Q be the orthonormal matrix (i.e. $Q^TQ = QQ^T = I$) whose columns are the corresponding normalized eigenvectors (i.e. $\|q_i\| = 1$). Then

$$A = Q\Lambda Q^T = \sum_{i=1}^{n} \lambda_i q_i q_i^T$$

This is called the eigenvalue decomposition or spectral decomposition of A.