Section 3
EE278: Introduction to Statistical Signal Processing (Fall 2020)

Monday, Oct 5, 2020 - 3 to 4 pm

1. Jointly Gaussian Random Variables

Definition: $\{Z_1, Z_2, \ldots, Z_n\}$ is a set of **jointly Gaussian zero-mean** random variables, and $\mathbf{Z} = (Z_1, \ldots, Z_n)^T$ is a **zero-mean Gaussian random vector**, if, for some finite set of IID $\mathcal{N}(0,1)$ random variables $W_1, \ldots, W_m$, each Z_j can be expressed as

$$Z_j = \sum_{l=1}^{m} a_{jl} W_l \quad \text{i.e., } \mathbf{Z} = A\mathbf{W}$$

where $\{a_{jl}, 1 \le j \le n, 1 \le l \le m\}$ are real numbers. More generally, $\mathbf{U} = (U_1, \ldots, U_n)^T$ is a Gaussian random vector if $\mathbf{U} = \mathbf{Z} + \boldsymbol{\mu}$ where $\mathbf{Z}$ is a zero-mean Gaussian random vector and $\boldsymbol{\mu}$ is a real vector.

Now show the following properties:

(a) If $\boldsymbol{Z}$ is a Gaussian random vector, then $\boldsymbol{Y} = A\boldsymbol{Z} + \boldsymbol{b}$ is also a Gaussian random vector, where A is a fixed matrix and $\boldsymbol{b}$ is a fixed vector.

(b) If $\boldsymbol{Z}$ is a Gaussian random vector, then the linear combination, i.e. $\boldsymbol{a}^T\boldsymbol{Z} = \sum_{i=1}^{n} a_i Z_i$ is a Gaussian random variable for any real vector $\boldsymbol{a}$. (What is its mean and variance?)

(c) If $\mathbf{U}$ and $\mathbf{V}$ are independent n-dimensional Gaussian random vectors, then

$$\mathbf{Y} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}$$

is a $2n$-dimensional random vector. Also $A\mathbf{U} + B\mathbf{V}$ is a Gaussian random vector for any matrices A and B.

Solution:

(a) By definition, $\boldsymbol{Z} = C\boldsymbol{W} + \boldsymbol{\mu}$ for some matrix C and vector $\boldsymbol{\mu}$, and $\boldsymbol{W}$ is a vector of IID $\mathcal{N}(0,1)$ random variables. Then $\boldsymbol{Y} = A\boldsymbol{Z} + \boldsymbol{b} = AC\boldsymbol{W} + A\boldsymbol{\mu} + \boldsymbol{b}$. Thus by definition, $\boldsymbol{Y}$ is also a Gaussian random vector. Further, $\boldsymbol{Y}$ has mean $A\boldsymbol{\mu} + \boldsymbol{b}$ and covariance matrix ACC^TA^T.

(b) As in (a), write $\boldsymbol{Z} = C\boldsymbol{W} + \boldsymbol{\mu}$. Then $\boldsymbol{a}^T\boldsymbol{Z} = \boldsymbol{a}^TC\boldsymbol{W} + \boldsymbol{a}^T\boldsymbol{\mu}$. Thus $\boldsymbol{a}^T\boldsymbol{Z}$ is a 1-d Gaussian random vector, which is just a Gaussian random variable. The mean of $\boldsymbol{a}^T\boldsymbol{Z}$ is $\boldsymbol{a}^T\boldsymbol{\mu}$ and the variance is $\boldsymbol{a}^TCC^T\boldsymbol{a} = \boldsymbol{a}^TK_{\boldsymbol{Z}}\boldsymbol{a}$.

Note: In some texts, you will find property (b) as the definition of a jointly Gaussian random variable, which is equivalent to the definition given above.

(c) By definition, we can write $\boldsymbol{U} = C_1\boldsymbol{W}_1 + \boldsymbol{\mu}_1$ and $\boldsymbol{V} = C_2\boldsymbol{W}_2 + \boldsymbol{\mu}_2$. Define the following:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}.$$

$\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are independent because $\boldsymbol{U}$ and $\boldsymbol{V}$ are independent. Thus $\boldsymbol{W}$ is a $2n$-vector of IID Gaussian random variables. We can write $\boldsymbol{Y} = C\boldsymbol{W} + \boldsymbol{\mu}$, hence $\boldsymbol{Y}$ is a $2n$-dimensional Gaussian random vector.

Finally, since $A\boldsymbol{U} + B\boldsymbol{V}$ is equal to $\boldsymbol{Y}$ left-multiplied by the matrix $[A\ B]$, use property (a) to say that $A\boldsymbol{U} + B\boldsymbol{V}$ is a Gaussian random vector. In fact $A\boldsymbol{U} + B\boldsymbol{V} + \boldsymbol{c}$ is a Gaussian random vector for any vector $\boldsymbol{c}$.

2. Examples (or not?) of Jointly Gaussian Random Variables

Let $Z_1 \sim \mathcal{N}(0,1)$ and let X be independent of Z_1 and take equiprobable values ± 1. Let $Z_2 = Z_1 X$

(a) What is the distribution of Z_2?

(b) What is $\mathbb{E}[Z_1 Z_2]$?

(c) Are Z_1 and Z_2 jointly Gaussian random variables? What is the joint pdf of (Z_1, Z_2)?

Solution:

(a) When $X = +1$, $Z_2 = Z_1$, and thus $Z_2 \sim \mathcal{N}(0,1)$. When $X = -1$, $Z_2 = -Z_1$, and again $Z_2 \sim \mathcal{N}(0,1)$. Thus $Z_2 \sim \mathcal{N}(0,1)$. We can formalize this as

$$\begin{aligned} f_{Z_2}(z) &= f_{Z_2}(z \mid X = +1)\Pr(X = +1) + f_{Z_2}(z \mid X = +1)\Pr(X = -1) \\ &= f_{Z_1}(z \mid X = +1) \cdot \frac{1}{2} + f_{Z_1}(-z \mid X = -1) \cdot \frac{1}{2} \\ &= \frac{1}{2}(f_{Z_1}(z) + f_{Z_1}(-z)) \\ &= f_{Z_1}(z) \end{aligned}$$

Thus Z_1 and Z_2 have the same marginal distributions, i.e. $\mathcal{N}(0,1)$.

(b) We can write

$$\begin{aligned} \mathbb{E}[Z_1 Z_2] &= \mathbb{E}[Z_1 Z_2 \mid X = +1]\Pr(X = +1) + \mathbb{E}[Z_1 Z_2 \mid X = -1]\Pr(X = -1) \\ &= \mathbb{E}[Z_1^2 \mid X = +1] \cdot \frac{1}{2} + \mathbb{E}[-Z_1^2 \mid X = -1] \cdot \frac{1}{2} \\ &= \frac{1}{2}(\mathbb{E}[Z_1^2] + \mathbb{E}[-Z_1^2]) \\ &= 0 \end{aligned}$$

Thus Z_1 and Z_2 are uncorrelated random variables.

(c) Although Z_1 and Z_2 are individually Gaussian random variables, they are not jointly Gaussian random variables. Consider the linear combination $Z_1 + Z_2$. This can be 0 with probability 1/2. Thus $Z_1 + Z_2$ is not a Gaussian random variable, hence violating property (b) in Question 1.

The joint pdf $f_{Z_1,Z_2}(z_1, z_2)$ is impulsive on the diagonals where $z_2 = \pm z_1$, and zero elsewhere.

Note: This example also shows the falseness of the frequently heard statement that uncorrelated Gaussian random variables are independent. The correct statement is that uncorrelated jointly Gaussian random variables are independent.

References

[1] R. G. Gallager. *Stochastic processes: theory for applications.* Cambridge University Press, 2013.