Section 4

EE278: Introduction to Statistical Signal Processing (Fall 2020)

Monday, Oct 12, 2020 - 3 to 4 pm

1. Detection in the presence of fading. Let the signal and observation, respectively, be

$$X = \begin{cases} -\sqrt{P} & \text{with probability } \frac{1}{2} \\ +\sqrt{P} & \text{with probability } \frac{1}{2}, \end{cases} \qquad \text{and} \qquad Y = AX + Z\,,$$

where $A \geq 0$ and $Z \sim \mathcal{N}(0, \sigma^2)$, and the random variables A, X and Z are independent.

(a) Suppose that A is random but known at the decoder. Find the optimal decoding rule $\hat{X}(y)$, i.e., the rule that minimizes the probability of decoding error.

(b) Now suppose that A is not known at the decoder but we know that $A \geq 0$. Find the optimal decoding rule $\hat{X}(y)$, i.e., the rule that minimizes the probability of decoding error.

Hint: How does the decoding rule in part (a) depend on A?

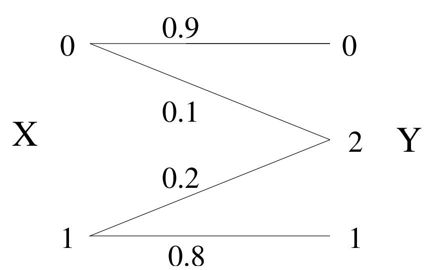
(c) Find an expression for the minimum probability of error in terms of signal power P, noise power σ^2, the pdf $f_A(a)$ of A, and the Q function.

Hint: Recall the error probability when $A = 1$. Use that to calculate the error probability conditioned on A. Then un-condition using $f_A(a)$.

(d) Suppose that instead of $A \geq 0$, A has a probability density symmetric about 0, i.e. $f_A(a) = f_A(-a)$. What is the optimal decoder and what is the minimum probability of error?

(e) Repeat part (d) when $X = +\sqrt{P}$ with probability p_+?

2. Binary erasure channel. The communication channel shown in the figure below has binary input $X \sim \text{Bern}(1/3)$ and ternary output $Y \in \{0, 1, 2\}$.



The conditional pmf $p_{Y|X}(y|x)$ of Y given X is given in the figure; e.g., $p_{Y|X}(2|0) = 0.1$.

1. Find $p_{X,Y}(x, y)$, $p_Y(y)$, and $p_{X|Y}(x|y)$ for $y = 0, 1, 2$.
2. A decoder $\hat{X}(y) \in \{0, 1\}$ decides whether X is 0 or 1 from the observed output Y. Specify the $\hat{X}(y)$ that minimizes the probability of error $P_e = \Pr\{\hat{X}(Y) \neq X\}$.
3. Find the minimum error probability P_e which is achieved by the optimal decoder.

3. Poisson observations The parameter X is

$$X = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ 2 & \text{with probability } \frac{1}{2}. \end{cases}$$

The conditional pmf of the outputs $Y_1, ..., Y_k$ given $X = x$ is i.i.d. Poisson distribution with intensity x, i.e., $Y_1, ..., Y_k | X = x \overset{i.i.d.}{\sim} \text{Poisson}(x)$.

(a) Compute the MAP rule of estimating X based on $Y_1, ..., Y_k$. Identify a sufficient statistic.

Hint: If $Y \sim \text{Poisson}(x)$, then $\mathbb{P}(Y = y) = e^{-x} \frac{x^y}{y!}$ for $y = 0, 1, 2, \ldots$.

(b) Compute the error probability.

Hint: If $Y_i \overset{i.i.d.}{\sim} \text{Poisson}(x_i)$ for $i = 1, \ldots, k$, then $\sum_{i=1}^k Y_i \sim \text{Poisson}\left(\sum_{i=1}^k x_i\right)$