Section 5

EE278: Introduction to Statistical Signal Processing (Fall 2020)

Monday, Oct 19, 2020 - 3 to 4 pm

1. Sufficient statistics

Let $V = v(\mathbf{Y})$ be a function of $\mathbf{Y}$ for a binary hypothesis X with a discrete observation $\mathbf{Y}$. The following (for all sample values $\mathbf{y}$) are equivalent for $v(\mathbf{y})$ to be a sufficient statistic:

1. A function u exists such that $\Lambda(\mathbf{y}) = u(v(\mathbf{y}))$.

2. For any given positive a priori probabilities (i.e. p_X), the a posteriori probabilities satisfy

$$p_{X|\mathbf{Y}}(x \mid \mathbf{y}) = p_{X|\mathbf{Y},V}(x \mid \mathbf{y}, v(\mathbf{y})) = p_{X|V}(x \mid v(\mathbf{y}))$$

   This means that $X - V - \mathbf{Y}$ is a Markov chain, i.e. X and $\mathbf{Y}$ are conditionally independent given V.

3. The likelihood ratio of $\mathbf{y}$ is the same as that of $v(\mathbf{y})$, i.e.

$$\Lambda(\mathbf{y}) = \frac{p_{V|X}(v(\mathbf{y}) \mid 1)}{p_{V|X}(v(\mathbf{y}) \mid 0)}$$

2. Whitening the Noise

1. Let $\mathbf{Z} = A\mathbf{W}$ where $\mathbf{W} \sim \mathcal{N}(0, I)$ is normalized IID Gaussian and A is non-singular. The observation random vector $\mathbf{Y}$ is $\mathbf{a} + \mathbf{Z}$ given $X = \mathbf{a}$, and $\mathbf{b} + \mathbf{Z}$ given $X = \mathbf{b}$. Suppose the observed sample $\mathbf{y}$ is transformed into $\mathbf{v} = A^{-1}\mathbf{y}$. Explain why $\mathbf{v}$ is a sufficient statistic for this detection problem (and thus why MAP detection based on $\mathbf{v}$ must yield the same decision as that based on $\mathbf{y}$).

2. Consider the detection problem where $\mathbf{V} = A^{-1}\mathbf{a} + \mathbf{W}$ given $X = \mathbf{a}$ and $\mathbf{V} = A^{-1}\mathbf{b} + \mathbf{W}$ given $X = \mathbf{b}$. Find $LLR(\mathbf{v})$ for a sample value $\mathbf{v}$ of $\mathbf{V}$. Show that this is the same as the LLR for a sample value $\mathbf{y} = A\mathbf{v}$ of $\mathbf{Y}$.

3. Find $\Pr\{e \mid X = \mathbf{a}\}$ and $\Pr\{e \mid X = \mathbf{b}\}$ for the detection problem in (b). Show that you get the same answer by evaluating using the conditional densities from (a) directly.

   Note: The methodology here is to transform the observed sample value to make the noise IID; this approach is often both useful and insightful.

References

[1] R. G. Gallager. *Stochastic processes: theory for applications.* Cambridge University Press, 2013.