Section 6

EE278: Introduction to Statistical Signal Processing (Fall 2020)

Monday, Oct 26, 2020 - 3 to 4 pm

1. Singular Covariance Matrix. If $X, \mathbf{Y}$ are jointly Gaussian, the MMSE estimator of X given vector $\mathbf{Y}$ is given by:

$$\hat{X}(\mathbf{Y}) = K_{X\mathbf{Y}} K_{\mathbf{Y}}^{-1} (\mathbf{Y} - \bar{\mathbf{Y}}) + \bar{X}, \tag{1}$$

where $\bar{\mathbf{Y}}$ and $\bar{X}$ are the mean of $\mathbf{Y}$ and X.

However, this formula would not work if the covariance matrix $K_{\mathbf{Y}}$ is not invertible. In this problem, you are required to investigate this situation and obtain the corresponding solutions.

(a) Prove that if the covariance matrix of a d-dimensional random vector $\mathbf{Y}$ is not invertible, then there must exist at least one component of $\mathbf{Y}$ which is expressible as a linear combination of the others.

(b) Suppose $X \sim \mathcal{N}(0, \sigma_X^2)$ (a scalar random variable) and $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, K_{\mathbf{Y}})$ (a random vector) are jointly Gaussian, where

$$K_{\mathbf{Y}} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix}, \tag{2}$$

and the covariance between X and $\mathbf{Y}$ is

$$K_{X\mathbf{Y}} = [3, 4, 7]. \tag{3}$$

Derive the MMSE estimator of X based on $\mathbf{Y}$.

2. Noise cancellation A classic problem in statistical signal processing involves estimating a weak signal (e.g., the heart beat of a fetus) in the presence of a strong interference (the heart beat of its mother) by making two observations—one with the weak signal present and one without (by placing one microphone on the mother's belly and another close to her heart). The observations can then be combined to estimate the weak signal by "canceling out" the interference. The following is a simple version of this application.

Let the weak signal $X \sim \mathcal{N}(\mu, P)$ Let the observations be $Y_1 = X + Z_1$ and $Y_2 = Z_1 + Z_2$, where Z_1 is the strong interference and Z_2 is measurement noise. Assume that $Z_1 \sim \mathcal{N}(0, N_1)$ and $Z_2 \sim \mathcal{N}(0, N_2)$ Further assume that X, Z_1, and Z_2 are independent. Find the MMSE estimate of X given Y_1 and Y_2, and the corresponding MSE. Interpret the results.