Chapter 15

Wind Energy

Problem Solutions

Prob 15.1 At a given sea-level location, the wind statistics, taken over a period of one year and measured at an anemometer height of 10 m above ground, are as follows:

Number of hours	**Velocity** (m/s)
90	25
600	20
1600	15
2200	10
2700	5
remaining time	calm

The velocity is to be assumed constant in each indicated range (to simplify the problem).

Although the wind varies with the 1/7th power of height, assume that the velocity the windmill sees is that at its center.

The windmill characteristics are:

Efficiency	70%
Cost	150 $\$/m^2$
Weight	100 kg/m^2

The areas mentioned above are the swept areas of the windmill.

If h is the height of the tower and M is the mass of the windmill on top of the tower, then the cost, C_T, of the tower is

$$C_T = 0.05\ h\ M.$$

How big must the swept area of the windmill be so that the average delivered power is 10 kW? How big is the peak power delivered, i.e., the rated power of the generator? Be sure to place the windmill at the most economic height. How tall will the tower be?

Neglecting operating costs and assuming an 18% yearly cost of the capital invested, what is the cost of the MWh?

If the windmill were installed in La Paz, Bolivia, a city located at an altitude of 4000 m, what would the average power be, assuming that the winds had the velocity given in the table above? The scale height of the atmosphere is 8000 meters (i.e., the air pressure falls exponentially with height with a characteristic length of 8000 m).

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The first step in the solution of this problem is to find what the mean wind velocity is at the anemometer height (10 m, in this problem). The significant mean is the *mean cubic wind velocity*, $< v >$, because the power in the wind is proportional to the cube of the velocity. The statistics are

Solution of Problem 15.1

given in hours. Since there are 8760 hours in a year, 90 hours, for instance, correspond to 90/8760 of a year and so on.

$$< v >\equiv \left(\frac{1}{T}\int_0^T v^3 dt\right)^{1/3} \rightarrow \left(\frac{1}{T}\sum_i v_i^3 \Delta t\right)^{1/3}$$

$$= \left[\frac{90\times 25^3 + 600\times 20^3 + 1600\times 15^3 + 2200\times 10^3 + 2700\times 5^3}{8760}\right]^{1/3}$$

$$= 11.7 \text{ m/s}$$

Let
C be the cost of the complete plant (windmill plus tower),
$C_W = 150A_v$ be the cost of the windmill,
$C_T = 0.05h \times 100A_v$ be the cost of the tower.
The cost of land, etc. has been neglected.

$$C = C_W + C_T$$

The wind velocity increases with height according to

$$< v >=< v_0 > \left(\frac{h}{h_0}\right)^{1/7}$$

where v_0 is the velocity at h_0 (10 m).
The mean generated power is

$$\langle P_g\rangle = \frac{16}{27}\times\frac{1}{2}\rho < v_0 >^3 \left(\frac{h}{h_0}\right)^{3/7} \eta A_v$$

$$= \frac{16}{27}\times\frac{1}{2}\times 1.29\times 11.7^3\times\left(\frac{1}{10}\right)^{3/7}\times 0.7\times h^{3/7}A_v$$

$$= 160\ h^{3/7}A_v \quad \text{W}$$

Let c be the cost of the plant in dollars per watt:

$$c \equiv \frac{C}{P_g} = \frac{C_W + C_T}{P_g} = \frac{150Av + 0.05h\times 100A_v}{160h^{3/7}A_v} = \frac{150+5h}{160h^{3/7}},$$

$$\frac{d\beta}{dh} = -\frac{3}{7}\times\frac{150+5h}{160h^{10/7}} + \frac{5}{160h^{3/7}} = 0$$

$$3\frac{150+5h}{h} = 35$$

$$h_{optimum} = 22.5 \text{ m}$$

Solution of Problem 15.1

Optimum tower height is 22.5 m.

At $h_{optimum}$, $\langle P_g \rangle = 160 \times 22.5^{3/7} = 608$ Wm^{-2}
For 10 kW, $A_v = 10^4/608 = 16.5$ m^2.

Swept area must be 16.5 m^2.

$C = 150 \times 16.5 + 0.05 \times 100 \times 16.5 \times 22.5 = \$4330$.
Annual cost of investment: $c = 0.18 \times 4330 = \$780$ yr^{-1}
Energy generated annually: $W_g = 10^4 \times 3.17 \times 10^7 = 3.17 \times 10^{11}$ J yr^{-1} where 3.17×10^7 is the number of seconds in a year.

$$W_g = \frac{3.17 \times 10^{11} \text{ J yr}^{-1}}{3600 \text{ J/Wh}} = 88 \times 10^6 \quad \text{Wh per year.}$$

Cost of electricity:

$$c_E = \frac{780 \text{ \$ yr}^{-1}}{88 \text{ MWh yr}^{-1}} = \text{\$8.86 per MWh or 8.86 mils per kWh.}$$

Cost of the generated electricity is \$8.86/MWh.

PG&E rates are about 4 mills per kWh.
Peak power $\equiv P = 10 \times (2.5/11.7)^3 = 97.6$ kW.

The generator must be able to handle 97.6 kW.

Thus, the generator must have a *rated* power of about 100 kW to deliver an *average* power of 10 kW. This is one of the difficulties of wind energy: the ratio of rated to average is much larger than in a hydroelectric plant where it may be as low as 2:1.

In Bolivia, the air density is much smaller because it decays exponentially with altitude with a "scale height", H, (a characteristic height) of some 8000 m:

$$\rho = \rho_0 \exp -\frac{h - h_0}{H} = 1.29 \exp -\frac{4000 - 10}{8000} = 0.78 \text{ kg m}^{-3},$$

$$< P_g >= 10 \frac{0.78}{1.29} = 6 \text{ kW}.$$

At La Paz, the average generated power would be 6 kW.

Solution of Problem 15.1

Prob 15.2 A utilities company has a hydroelectric power plant equipped with generators totaling 1 GW capacity. The utilization factor used to be exactly 50%, i.e., the plant used to deliver every year exactly half of the energy the generators could produce. In other words, the river that feeds the plant reservoir was able to sustain exactly the above amount of energy. During the wet season, the reservoir filled but never overflowed.

Assume that the plant head is an average of 80 m and that the plant (turbines and generators) has an efficiency of 97%.

What is the mean rate of flow of the river (in m^3/s)?

With the industrial development of the region, the utilities company wants to increase the plant utilization factor to 51% but, of course, there is not enough water for this. So, they decided to use windmills to pump water up from the level of the hydraulic turbine outlet to the reservoir (up 80 m).

A careful survey reveals that the wind regime is the one given in the table below:

$< v >$ (m/s)	Θ
5	0.15
7	0.45
10	0.30
12	0.10

$<v>$ is the mean cubic wind velocity and θ is the percentage of time during which a given value of $< v >$ is observed. The generator can be dimensioned to deliver full power when $<v>= 12$ m/s or when $<v>$ is smaller. If the generator is chosen so that it delivers its rated full power for, say, $<v>= 10$ m/s, then a control mechanism will restrict the windmill to deliver this power even if $<v>$ exceeds the 10 m/s value.

Knowing that the cost of the windmill is \$10 per m^2 of swept area, that of the generator is \$0.05 per W of rated output, the efficiency of the windmill is 0.7 and that of the generator is 0.95, calculate which is the most economic limiting wind velocity.

What is the swept area of the windmill that will allow increasing the plant factor to 51% ? (The pumps are 95% efficient). What is the cost of the MWh generated by the windmills, assuming an annual cost of investment of 20% and neither maintenance nor operating costs.

The windmills are, essentially, at sea level.

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Solution of Problem 15.2

With an utilization factor of 50%, the 1 GW (peak) power plant generates 0.5×10^9 W, on the average. Owing to the 97% efficiency of the plant, the water power required (average) is $0.5 \times 10^9/0.97 = 0.515 \times 10^9$ W.

Let $\dot{M}$ be the rate of water use by the turbines (in kg/s). Then, if the head is h, the water power is $\dot{M}gh$, where g is the acceleration of gravity.

$$\dot{M} = \frac{0.515 \times 10^9}{9.81 \times 80} = 657 \times 10^3 \quad \text{kgs}^{-1} \qquad \text{or} \quad 657 \quad \text{m}^3\text{s}^{-1}.$$

The mean rate of flow of the river is 657 m^3s^{-1}.

To increase the plant utilization factor by 2%, the above average flow rate must be raised to 670.1 m^3s^{-1}, i.e., 13.1 m^3s^{-1} of water must be pumped back up into the reservoir. The required pumping power is

$$P_{pump} = \frac{1}{0.95} 13.1 \times 10^3 \times 9.81 \times 80 = 10.8 \times 10^6 \qquad \text{W}.$$

Let v be the wind velocity and v_m the "limiting wind velocity", i.e. the wind velocity above which the windmill generates the same amount as energy as at v_m.

As long as $v \le v_m$, the power generated by the windmill is

$$P_g = \frac{16}{27} \times \tfrac{1}{2}\rho v^3 \eta_w \eta_g A_v = \frac{16}{27} \times \tfrac{1}{2} \times 1.29 \times 0.7 \times 0.95 v^3 A_v = 0.254 v^3 A_v.$$

If the limiting velocity is $< v_k >$ (the kth value in the wind distribution table), then the energy generated in a year is

$$W = \underbrace{8760}_{hours/year} \times 0.254 A_v \underbrace{\left[\sum_{i=1}^{k} < v_i >^3 \Theta_i + < v_k >^3 \sum_{i=k+1}^{N} \Theta_i\right]}_{\equiv V_{mean}^3}.$$

$$W = 2227 A_v V_{mean}^3. \qquad \text{W hr yr}^{-1}$$

N is the total number of entries in the Table (4, in this problem). The table lists the value of V_{mean}^3 for each case.

k	v_k (ms¹)	A: $\sum_{i=1}^{k} <v_i>^3 \Theta_i$	B: $<v_k>^3 \sum_{i=k+1}^{N} \Theta_i$	$V_{mean}^3 = A + B$
1	5	$0.15\times5^3=18.75$	$(0.45+0.30+0.10)\times5^3= =106.25$	125.0
2	7	$0.15\times5^3+0.45\times7^3= =173.1$	$(0.30+0.10)7^3= =137.2$	310.3
3	10	$0.15\times5^3+0.45\times7^3+ +0.3\times10^3=473.1$	$0.1\times10^3= =100$	573.1
4	12	$0.15\times5^3+0.45\times7^3+ +0.3\times10^3+0.1\times12^3= =645.9$		645.9

Solution of Problem 15.2

We saw that, averaged over one year, the wind turbines have to generate 10.8 MW or 94,610 MWh over the whole year. We also saw that the power generated by the wind turbine is $P_g = 0.254 < V >^3 A_v$ and, thus the energy generated over one year period is $W_g = 2227, V >^3 A_v = 94.61 \times 10^9$ Wh. From this,

$$A_v = \frac{42.49 \times 10^6}{< V >^3} \quad \text{(Column 3 of the table below).}$$

The cost of the wind turbine is \$10/m^2, hence

$$C_w = 10A_v \quad \text{(Column 4).}$$

The rated power of the generator is

$$P_g = V_{limit}^3 \times A_v \times 0.254 \quad \text{(Column 5).}$$

The cost of the generator is

$$C_g = 0.05P_g \quad \text{(Column 6).}$$

The total cost is

$$C_t = C_w + C_g \quad \text{(Column 7).}$$

The total yearly cost is

$$c = 0.2C_t. \quad \text{(Column 8).}$$

The cost per MWh is

$$c/94610. \quad \text{(Column 9).}$$

1	2	3	4	5	6	7	8	9
Lim.	$<V>^3$	A_v	C_w	P_g	C_g	C_t	c	c
Vel.		$\frac{42.49\times10^6}{<V>^3}$	$10A_v$	$0.254V_k^3A_v$	$0.05P_g$	C_w+C_g	$0.2C_t$	c/94610
(m/s)		(m^2)	(M\$)	(MW)	(M\$)	(M\$)	(M\$)	(\$/MWh)
5	125	339,920	3.399	10.8	0.540	3.939	0.788	8.326
7	310.1	136,932	1,369	11.9	0.596	1.966	0.393	4.156
10	573.1	74,141	0.741	18.8	0.942	1.683	0.336	3.558
12	645.9	65,784	0.658	28.9	1.444	2.102	0.420	4.442

The minimum cost of electricity occurs for $k = 3$.
It is 3.558 \$ per MW h.
Since 10.8 MW are needed to pump water,

$$P_g = 0.254\ V_{mean}^3 A_v,$$

$$A_v = \frac{10.8 \times 10^6}{254 \times 573} = 74,200 \quad \text{m}^2.$$

The overall swept area is 74,000 square meters. It is possible that several windmills would be used. Four Boeing Model 2 would do.

Solution of Problem 15.2

Prob 15.3 A windmill is installed at a sea-level site where the wind has the following statistics:

v (m/s)	% of time
0	30
3	30
9	30
12	8
15	2

The velocity in the table above should be the mean cubic velocity of the wind, however, to simplify the problem, assume that the wind actually blows at a constant 3 m/s 30% of the time, a constant 9 m/s another 30% of the time and so on.

The windmill characteristics are:

Efficiency (including generator)	0.8
Windmill cost	200 $/m^2 of swept area
Generator cost	200 $/kW of rated power.

Rated power is the maximum continuous power that the generator is supposed to deliver without overheating. The duty cycle is 1, i.e., the windmill operates continuously (when there is wind) throughout the year. Consider only investment costs. These amount to 20% of the investment, per year.

The system can be designed so that the generator will deliver full (rated) power when the wind speed is 15 m/s. The design can be changed so that rated power is delivered when the wind speed is 12 m/s. In this latter case, if the wind exceeds 12 m/s, the windmill is shut down. It also can be designed for rated power at 9 m/s, and so on.

We want a windmill that delivers a maximum of 1 MW. It has to be designed so that the cost of the generated electricity over a whole year is minimized. What is the required swept area? What is the cost of electricity?

The power generated by the windmill is

$$P_g = \frac{16}{27} \times \tfrac{1}{2}\rho\eta A_v V^3 = 0.306 A_v V^3,$$

where A_v is the swept area of the windmill.

The generator is to be rated at 1 MW at a chosen **maximum rated wind velocity,** V_{rated}. Consequently, the swept area required is a function

Solution of Problem 15.3

of this V_{rated}:

$$A_v = \frac{10^6}{0.306} \times \frac{1}{V_{rated}^3} = \frac{3.27 \times 10^6}{V_{rated}^3}.$$

According to the problem we must choose different maximum rated wind velocities and for each we must compute the swept area so that we can find the windmill cost, C_w, which is \$200/kW. Since the rated power of the generator is 1 MW in all cases, the generator will cost \$200,000 or 200 k\$ in all cases.

We can now construct the table below:

Max. rated wind vel., V_{rated}	3	9	12	15	m/s
Required swept area, A_v	121,000	4,486	1,893	969	m^2
Windmill cost, C_w	24,200	897	378	194	k$
Generator cost, C_g	200	200	200	200	k$
Total investment, $C_w + C_g$	24,400	1,097	578	394	k$
Yearly cost of investment, $0.2(C_w + C_g)$	4,880	219.4	115.6	78.8	k$

Naturally, the yearly investment cost for a fixed maximum electric power falls rapidly when the windmill is designed to operate with larger wind velocities because it then requires smaller swept areas. However, the total amount of energy delivered in a year also falls with increasing maximum design wind velocities. The reason is that most of the time the winds are below maximum and are not used effectively when the swept areas are small.

The energy generated during the year is

$$W = \underbrace{8760}_{hours/year} < P_g > \qquad \text{Wh},$$

$$W = 0.306 \times 8760 \times [0.3 \times 3^3 + 0.3 \times 9^3 + 0.08 \times 12^3 + 0.2 \times 15^3] A_v$$
$$= 2680[8.1 + 218.7 + 138.2 + 67.5] A_v \qquad \text{Wh},$$

Truncate after	3	9	12	15	m/s
W	2629	2729	1853	1124	MW h^2
Cost of electricity	1856	80.4	62.4	70.1	\$/MW h

The most economical electricity results from causing the windmill to cut off when the wind velocity exceeds 12 m/s.

Solution of Problem 15.3

Prob 15.4 *For this problem, you need a programmable calculator or a computer*

Consider an airfoil for which

$C_L = 0.15\alpha$**,**

$C_D = 0.015 + 0.015|\alpha|$,

for $-15° < \alpha < 15°$ **where** α **is the angle of attack. A wing with this airfoil is used in a vertical-axis windmill having a radius of 10 m. The setup angle is 0.**

Tabulate and plot α **as well as the quantity** D **(see text) as a function of** θ**, for** $U/V = 6$**. Use increments of** θ **of 30° (i.e., calculate 12 values).**

Considerations of symmetry facilitate the work. Be careful with the correct signs of angles. It is easy to be trapped in a sign error. Find the mean value of $< D >$**.**

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Let us calculate $< D >$ for a given value of U/V.

From the Text,

$$D \equiv \Gamma^2 (C_L \sin\psi - C_D cos\psi). \tag{1}$$

But C_L and C_D are functions of the angle of attack, α:

$$\alpha = \psi + \xi = \psi,$$

because, in this problem, $\xi = 0$.

$$C_L = 0.15\psi, \tag{2}$$

$$C_D = 0.015 + |0.015\psi|. \tag{3}$$

We need ψ and Γ as a function of the angular position, θ.

$$\Gamma^2 = 1 + \frac{U^2}{V^2} + 2\frac{U}{V}\sin\theta. \tag{4}$$

Since $\frac{U}{V} = 6$ (given in the problem),

$$\Gamma^2 = 37 + 12\sin\theta. \tag{5}$$

and

$$\cos\psi = \frac{U/V + \sin\theta}{\Gamma} = \frac{6 + \sin\theta}{\sqrt{37 + 12\sin\theta}}. \tag{6}$$

$$\psi = \arccos(\cos\psi). \tag{7}$$

Introducing Equations 2, 3 and 4 into Equation 1,

$$D = (37 + 12\sin\theta)[0.15\psi\sin\psi - (0.015 + |0.015\psi|)\cos\psi]$$

which, through Equations 6 and 7, is a function of θ alone.

Evaluation of D for selected values of θ must be done numerically. A spread sheet program such as Excel is particularly suitable for this purpose. See the printout on next page.

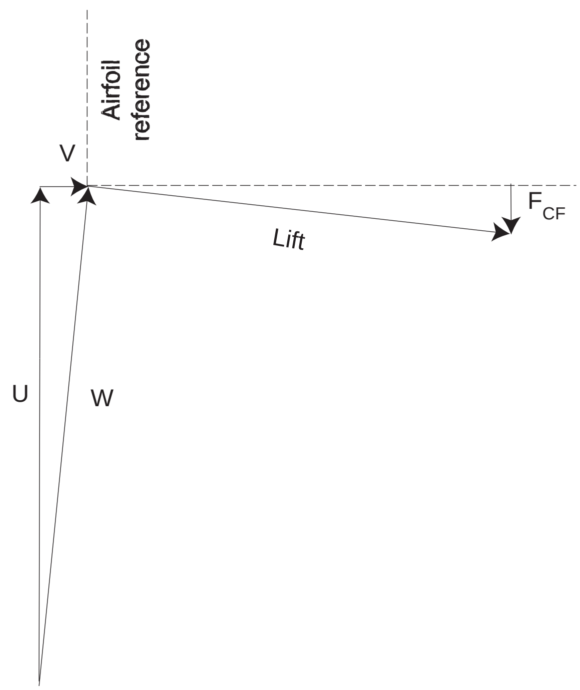
Solution of Problem 15.4

θ deg	Γ	α deg	C_L	C_D	F_{CF}	F_{CB}	T_F	D	D
0	6.0828	9.4623	1.4193	0.1569	0.2333	0.1548	0.0785	2.9060	2.9060
5	6.1681	9.2943	1.3942	0.1544	0.2252	0.1524	0.0728	2.7689	
10	6.2517	9.0633	1.3595	0.1510	0.2142	0.1491	0.0651	2.5440	
15	6.3329	8.7732	1.3160	0.1466	0.2007	0.1449	0.0558	2.2394	
20	6.4113	8.4281	1.2642	0.1414	0.1853	0.1399	0.0454	1.8662	
25	6.4862	8.0320	1.2048	0.1355	0.1683	0.1342	0.0342	1.4386	
30	6.5574	7.5890	1.1384	0.1288	0.1503	0.1277	0.0226	0.9732	0.9732
35	6.6244	7.1031	1.0655	0.1215	0.1318	0.1206	0.0111	0.4888	
40	6.6868	6.5782	0.9867	0.1137	0.1130	0.1129	0.0001	0.0052	
45	6.7443	6.0182	0.9027	0.1053	0.0946	0.1047	-0.0100	-0.4569	
50	6.7965	5.4269	0.8140	0.0964	0.0770	0.0960	-0.0190	-0.8769	
55	6.8432	4.8079	0.7212	0.0871	0.0604	0.0868	-0.0264	-1.2347	
60	6.8842	4.1650	0.6248	0.0775	0.0454	0.0773	-0.0319	-1.5116	-1.5116
65	6.9192	3.5017	0.5253	0.0675	0.0321	0.0674	-0.0353	-1.6909	
70	6.9481	2.8215	0.4232	0.0573	0.0208	0.0573	-0.0364	-1.7582	
75	6.9707	2.1278	0.3192	0.0469	0.0119	0.0469	-0.0350	-1.7024	
80	6.9870	1.4241	0.2136	0.0364	0.0053	0.0364	-0.0310	-1.5154	
85	6.9967	0.7137	0.1071	0.0257	0.0013	0.0257	-0.0244	-1.1930	
90	7.0000	0.0000	0.0000	0.0150	0.0000	0.0150	-0.0150	-0.7350	-0.7350
95	6.9967	-0.7137	-0.1071	0.0257	0.0013	0.0257	-0.0244	-1.1930	
100	6.9870	-1.4241	-0.2136	0.0364	0.0053	0.0364	-0.0310	-1.5154	
105	6.9707	-2.1278	-0.3192	0.0469	0.0119	0.0469	-0.0350	-1.7024	
110	6.9481	-2.8215	-0.4232	0.0573	0.0208	0.0573	-0.0364	-1.7582	
115	6.9192	-3.5017	-0.5253	0.0675	0.0321	0.0674	-0.0353	-1.6909	
120	6.8842	-4.1651	-0.6248	0.0775	0.0454	0.0773	-0.0319	-1.5116	-1.5116
125	6.8432	-4.808	-0.7212	0.0871	0.0604	0.0868	-0.0264	-1.2347	
130	6.7965	-5.4269	-0.8140	0.0964	0.0770	0.0960	-0.0190	-0.8769	
135	6.7443	-6.0183	-0.9027	0.1053	0.0946	0.1047	-0.0100	-0.4569	
140	6.6868	-6.5783	-0.9867	0.1137	0.1130	0.1129	0.0001	0.0052	
145	6.6244	-7.1032	-1.0655	0.1215	0.1318	0.1206	0.0111	0.4888	
150	6.5574	-7.5891	-1.1384	0.1288	0.1503	0.1277	0.0226	0.9732	0.9732
155	6.4862	-8.0321	-1.2048	0.1355	0.1683	0.1342	0.0342	1.4386	
160	6.4113	-8.4282	-1.2642	0.1414	0.1853	0.1399	0.0454	1.8662	
165	6.3329	-8.7733	-1.3160	0.1466	0.2007	0.1449	0.0558	2.2394	
170	6.2517	-9.0633	-1.3595	0.1510	0.2142	0.1491	0.0651	2.5440	
175	6.1681	-9.2944	-1.3942	0.1544	0.2252	0.1524	0.0728	2.7689	
180	6.0828	-9.4623	-1.4193	0.1569	0.2333	0.1548	0.0785	2.9060	2.9060
185	5.9962	-9.5634	-1.4345	0.1585	0.2383	0.1562	0.0821	2.9510	
190	5.909	-9.5938	-1.4391	0.1589	0.2398	0.1567	0.0832	2.9034	
195	5.8219	-9.5503	-1.4325	0.1583	0.2377	0.1561	0.0816	2.7663	
200	5.7355	-9.4298	-1.4145	0.1564	0.2317	0.1543	0.0774	2.5465	
205	5.6505	-9.2297	-1.3845	0.1534	0.2221	0.1515	0.0706	2.2541	
210	5.5678	-8.9483	-1.3422	0.1492	0.2088	0.1474	0.0614	1.9024	1.9024
215	5.4879	-8.5843	-1.2877	0.1438	0.1922	0.1422	0.0500	1.5073	
220	5.4117	-8.1377	-1.2207	0.1371	0.1728	0.1357	0.0371	1.0866	
225	5.3399	-7.6094	-1.1414	0.1291	0.1511	0.1280	0.0231	0.6599	
230	5.2733	-7.0015	-1.0502	0.1200	0.1280	0.1191	0.0089	0.2472	
235	5.2125	-6.3175	-0.9476	0.1098	0.1043	0.1091	-0.0048	-0.1310	
240	5.1583	-5.5625	-0.8344	0.0984	0.0809	0.0980	-0.0171	-0.4549	-0.4549
245	5.1112	-4.7429	-0.7114	0.0861	0.0588	0.0858	-0.0270	-0.7060	
250	5.0719	-3.8667	-0.5800	0.0730	0.0391	0.0728	-0.0337	-0.8674	
255	5.0407	-2.9432	-0.4415	0.0591	0.0227	0.0591	-0.0364	-0.9249	
260	5.0182	-1.9831	-0.2975	0.0447	0.0103	0.0447	-0.0344	-0.8669	
265	5.0046	-0.9979	-0.1497	0.0300	0.0026	0.0300	-0.0274	-0.6852	
270	5.0000	0.0000	0.0000	0.0150	0.0000	0.0150	-0.0150	-0.3750	-0.3750
275	5.0046	0.9978	0.1497	0.0300	0.0026	0.0300	-0.0274	-0.6852	
280	5.0182	1.9830	0.2975	0.0447	0.0103	0.0447	-0.0344	-0.8669	
285	5.0407	2.9431	0.4415	0.0591	0.0227	0.0591	-0.0364	-0.9249	
290	5.0719	3.8666	0.5800	0.0730	0.0391	0.0728	-0.0337	-0.8674	
295	5.1112	4.7429	0.7114	0.0861	0.0588	0.0858	-0.0270	-0.7060	
300	5.1583	5.5625	0.8344	0.0984	0.0809	0.0980	-0.0171	-0.4549	-0.4549
305	5.2125	6.3175	0.9476	0.1098	0.1043	0.1091	-0.0048	-0.1310	
310	5.2733	7.0014	1.0502	0.1200	0.1280	0.1191	0.0089	0.2472	
315	5.3399	7.6094	1.1414	0.1291	0.1511	0.1280	0.0231	0.6598	
320	5.4117	8.1377	1.2207	0.1371	0.1728	0.1357	0.0371	1.0866	
325	5.4879	8.5843	1.2876	0.1438	0.1922	0.1422	0.0500	1.5073	
330	5.5678	8.9482	1.3422	0.1492	0.2088	0.1474	0.0614	1.9024	1.9024
335	5.6505	9.2297	1.3845	0.1534	0.2221	0.1515	0.0706	2.2541	
340	5.7355	9.4297	1.4145	0.1564	0.2317	0.1543	0.0774	2.5465	
345	5.8219	9.5503	1.4325	0.1583	0.2377	0.1561	0.0816	2.7663	
350	5.9090	9.5938	1.4391	0.1589	0.2398	0.1567	0.0832	2.9034	
355	5.9962	9.5633	1.4345	0.1585	0.2383	0.1562	0.0821	2.9510	
							$<D>=$	**0.4701**	**0.5434**

Solution of Problem 15.4

In the printout of the preceding page:

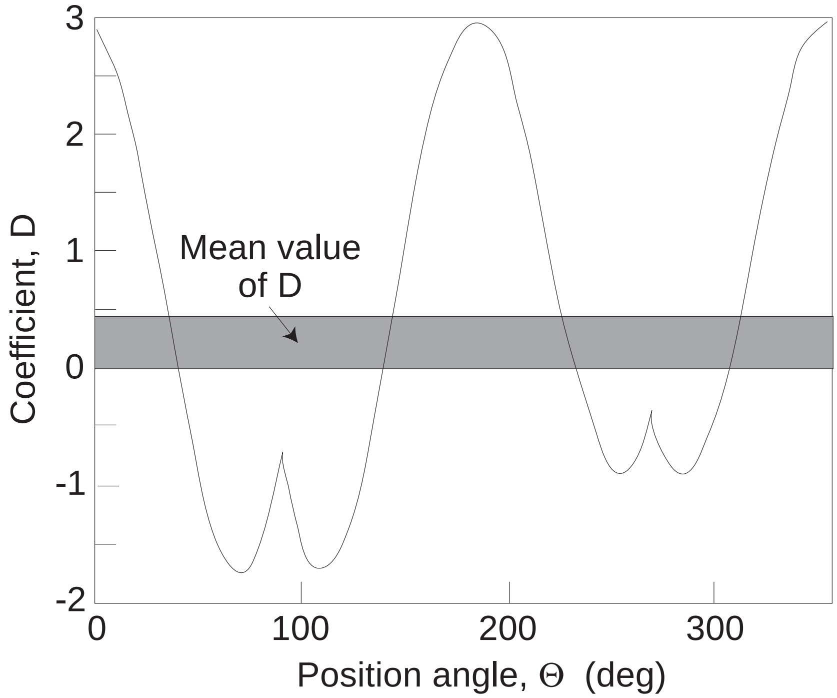
First column:	Sequence of position angles, θ.
Second column:	Values of Γ calculated from $\Gamma = \sqrt{37 + 12\sin\theta}$.
Third column:	Attack angle, $\alpha = \psi = \arccos \frac{6+\sin\theta}{\Gamma}$.
Fourth column:	$C_L = 0.15\alpha$.
Fifth column:	$C_D = 0.015 + \lvert 0.015\alpha \rvert$.
Sixth column:	$F_{CF} = C_L \sin\psi$.
Seventh column:	$F_{CB} = C_D \cos\psi$.
Eighth column:	Total torquing force, $T_F \equiv F_{CF} - F_{CB}$.
Ninth column:	Value of D for every 5° of θ.
Tenth column:	Value of D for every 30° of θ.



Various vectors when θ is 180°.

As can be seen from the figure above, although the angle of attack (the angle between the induced wind W and the reference plane of the airfoil) is negative in the second and third quadrants of θ, the lift force has a positive component (F_{CF}, i.e. it has a component that tends to torque the wing forward. This true in all four quadrants. There are, however sectors in which even with positive F_{CF}, the net torque is negative because the drag overwhelms the lift (between 45° and 135° and between 235° and 305°).

Solution of Problem 15.4



Plot of D as a function of the angular position of the wing.

Solution of Problem 15.4

Prob 15.5 In the region of Aeolia, on the island of Anemos, the wind has a most peculiar behavior. At precisely 0600, there is a short interval with absolutely no wind. Local peasants set their digital watches by this lull. Wind velocity then builds up linearly with time, reaching exactly 8 m/s at 2200. It then decays, again linearly, to the morning lull.

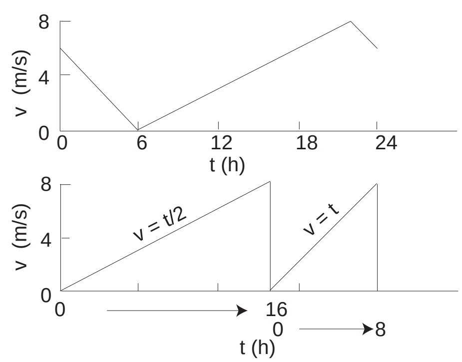
A vertical-axis windmill with 30 m high wings was installed in that region. The aspect ratio of the machine is 0.8 and its overall efficiency is 0.5. This includes the efficiency of the generator.

What is the average power generated? What is the peak power? Assuming a storage system with 100% turnaround efficiency, how much energy must be stored so that the system can deliver the average power continuously? During what hours of the day does the windmill charge the storage system?

Notice that the load always gets energy from the storage system. This is to simplify the solution of the problem. In practice, it would be better if the windmill fed the load directly and only the excess energy were stored.

....

To slightly simplify the integration we have to perform later on, we redrew the v vs t diagram of the top figure. The result is shown in the bottom figure. It consists of displacing the 0600 to 2200 wind speed rise so that it occurs between 0000 and 1600 and of reversing the 2200 to 0600 wind decline so that it looks like a speed rise from 0000 to 0800.



The mean cubic velocity, $< V^3 >$, is

$$< V^3 > = \frac{1}{T}\int_0^T V^3 dt = \frac{1}{24}\left[\int_0^{16}\left(\tfrac{1}{2}t\right)^3 dt + \int_0^8 t^3 dt\right]$$

$$= \frac{1}{24}\left[\left.\frac{t^4}{8\times 4}\right|_0^{16} + \left.\frac{t^4}{4}\right|_0^8\right] = 128 \quad \mathrm{m^3s^{-3}}.$$

Solution of Problem 15.5

The aspect ratio is

$$\Lambda = \frac{H}{2r} \quad \therefore \quad r = \frac{H}{2\Lambda} = \frac{30}{2 \times 0.8} = 18.75 \quad \text{m},$$

$$A_v = 2rH = 2 \times 18.75 \times 30 = 1125 \quad \text{m}^2.$$

The average power generated is

$$P_g = \frac{16}{27}\tfrac{1}{2}\rho < V^3 > A_v\eta = \frac{16}{27} \times \tfrac{1}{2} \times 1.29 \times 128 \times 1125 \times 0.5 = 27,500 \quad \text{W}.$$

To determine the peak power, we will use the cube of the peak wind velocity, $8^3 = 512\ \text{m}^3\text{s}^{-3}$ instead of the cubic mean velocity of $128\ \text{m}^3\text{s}^{-3}$, hence,

$$P_{g_{peak}} = 27,500 \times \frac{512}{128} = 110,000 \quad \text{W}.$$

The windmill generates excess energy whenever the wind velocity, V, exceeds the cube root of $< V^3 >$ which is 5.04 m/s.

Between 0600 and 2200, the wind velocity is $V = -3 + \frac{1}{2}t$.

$$5.04 = -3 + \tfrac{1}{2}t \quad \therefore \quad t = 16.08 \text{ or } 16:05.$$

Between 2200 and 0600, $V = 30 - t$. (0600 was taken as 3000).

$$5.04 = 30 - t \quad \therefore \quad t = 24.96 \text{ or } 00.96 \text{ or } 00:57.$$

Between 16:05 and 00:57, the windmill generates more energy than the loads uses. The energy generated by the windmill during this period is

$$W = \frac{16}{27}\tfrac{1}{2}\rho A_v \eta \left[\int_{16.08}^{22} (-3 + \tfrac{1}{2}t)^3 dt + \int_{22}^{24.96} (30 - t)^3 dt\right] \times 3600.$$

The factor, 3600, is the number of seconds in an hour.

$$W = 6 \times 10^5 \left\{ \int_{16.08}^{22} \left(-27 + \frac{27}{2}t - \frac{9}{4}t^2 + \frac{1}{8}t^3\right) dt \right.$$
$$\left. + \int_{22}^{24.96} (2700 - 2700t + 90t^2 - t^3) dt \right\}$$
$$= 7.74 \times 10^5 \left\{ \left[-27t + \frac{27}{2}t^2 - \frac{9}{12}t^3 + \frac{1}{32}t^4\right]_{16.08}^{22} \right.$$
$$\left. + \left[27000t - \frac{2700}{2}t^2 + \frac{90}{3}t^3 - \frac{1}{4}t^4\right]_{22}^{24.96} \right\},$$

Solution of Problem 15.5

$$W = 7.74 \times 10^5 \times 2588 = 2 \times 10^9 \quad \text{J.}$$

During the 16:05 to 00:57 time period (a total of 31,970 seconds), the energy taken by the load is $27{,}500 \times 31,970 = 0.879 \times 10^9$ J. Hence, a total of $2 - 0.897 = 1.12$ GJ (311 kWh) must be stored.

The average power generated is 27.5 kW.

The peak power generated is 110 kW.

The energy to be stored is 311 kWh.

The storage system is charged between 16:05 and 0:57.

Solution of Problem 15.5

Prob 15.6 A vertical-axis windmill with a rectangular swept area has an efficiency whose dependence on the U/V ratio (over the range of interest) is given, approximately, by

$$\eta = 0.5 - \frac{1}{18}\left(\frac{U}{V} - 5\right)^2.$$

The swept area is 10 m^2 and the aspect ratio is 0.8.

The wind velocity is 40 km/h.

What is the maximum torque the windmill can deliver? What is the number of rotations per minute at this torque? What is the power delivered at this torque? What is the radius of the windmill and what is the height of the wings? If the windmill drives a load whose torque is given by

$$\Upsilon_L = \frac{1200}{\omega} \qquad \text{Nm}$$

where ω is the angular velocity, what is the power delivered to the load? What are the rpm when this power is being delivered?

The dimensions, r and H, of the windmill are needed to solve the problem. They are given indirectly in the statement that specifies the swept area, A_v, and the aspect ratio, Λ.

$$\Lambda = \frac{A_v}{4r^2} \ \therefore\ r = \sqrt{\frac{A_v}{4\Lambda}} = \sqrt{\frac{10}{4 \times 0.8}} = 1.77 \qquad \text{m.}$$

The aspect ratio is also given by

$$\Lambda = \frac{H}{2r} \ \therefore\ H = 2r\Lambda = 2 \times 1.77 \times 0.8 = 2.83 \qquad \text{m.}$$

40 km/h is the same as 11.1 m/s.

The power delivered by the windmill is

$$P_D = \frac{16}{27}\tfrac{1}{2}\rho V^3 A_v \eta = 5243\eta.$$

We have used the value of 1.29 kg/m^3 for the air density.

$$\omega = \frac{U}{r} = \frac{U}{V}\frac{V}{r} \ \therefore\ \frac{U}{V} = \frac{\omega r}{V}.$$

The torque, Υ, is

$$\Upsilon = \frac{P_D}{\omega} = \frac{5243}{\omega}\left|0.5 - \frac{1}{18}\left(\frac{U}{V} - 5\right)^2\right| = \frac{5243}{\omega}\left|0.5 - \frac{1}{18}\left(\frac{\omega r}{V} - 5\right)^2\right|.$$

Solution of Problem 15.6

Introducing the known values of r and V into the preceding equation,

$$\Upsilon = -7.393\omega + 464.1 - 4660/\omega.$$

The torque reaches an extremum (in this case a maximum) when

$$\frac{d\Upsilon}{d\omega} = \frac{4660}{\omega^2} - 7.393 = 0 \;\therefore\; \omega = \sqrt{\frac{4660}{7.393}} = 25.1 \qquad \text{rad/s.}$$

It should be noted that the maximum torque does not occur when the efficiency is maximum. From inspection, it can be seen that the maximum efficiency is 0.5. Maximum torque occurs when $U = r\omega = 44.4$ m/s and U/V is $44.44/11.1 = 4.0$. This leads to an efficiency of 0.444.

The number of rotations per minute is

$$\text{rpm} = 25.1\frac{60}{2\pi} = 240.$$

Introducing the value of ω into the expression for torque, we obtain

$$\Upsilon_{max} = 92.9 \qquad \text{N m.}$$

$$P_{max} = \omega\Upsilon_{max} = 25.1 \times 92.9 = 2331 \qquad \text{W.}$$

The torque versus angular velocity characteristic of the load is

$$\Upsilon_L = \frac{1200}{\omega} \qquad \text{N m.}$$

The windmill will operate at the point at which its torque equals that of the load:

$$\frac{1200}{\omega} = -\frac{4660}{\omega} + 464.1 - 7.393\omega,$$

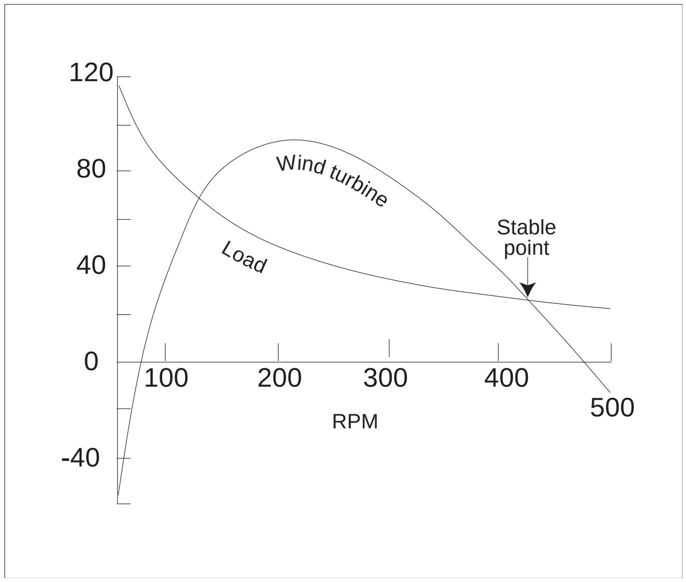
$$7.393\omega^2 - 464.1\omega + 5860 = 0,$$

$$\omega = \frac{464.1 \pm \sqrt{464.1^2 - 4 \times 7.393 \times 5860}}{2 \times 7.393}$$

$$= \frac{464.1 \pm 205.1}{14.786} = \begin{cases} 45.3 & \text{rad/sec} \quad (433 \text{ rpm}), \\ 17.5 & \text{rad/sec} \quad (167 \text{ rpm}). \end{cases}$$

Of the two operating points, only the one at the higher rpm is stable, as discussed in the Text. The power delivered is, of course, 1200 W because the load works at constant power independently of the rpm.

Solution of Problem 15.6



The maximum torque the windmill can deliver is 92.9 N m.

At maximum torque, the windmill turns at 240 rpm.

At maximum torque, the windmill can deliver 2331 W.

The radius of the windmill is 1.77 m.

The height of the wings is 2.83 m.

To the load, the windmill delivers 1200 W at 433 rpm.

Solution of Problem 15.6

Prob 15.7 An engineering firm has been asked to make a preliminary study of the possibilities of economically generating electricity from the winds in northeastern Brazil. As a first step, a quick and very rough estimate is required. Assume that the efficiency can be expressed by the ultra-simplified formula of Equation 48 in the text. The results will be grossly over-optimistic because we fail to take into account a large number of loss mechanisms and we assume that the turbine will always operate at the best value of $\frac{U}{V}D$ which, for the airfoil in question is 4.38. Our first cut will lead to unrealistic results but will yield a ball park idea of the quantities involved.

In the region under consideration, the trade winds blow with amazing constancy, at 14 knots. Assume, that this means 14 knots at a 3 m anemometer height. The wind turbines to be employed are to have a rated power of 1 MWe (MWe = MW of electricity).

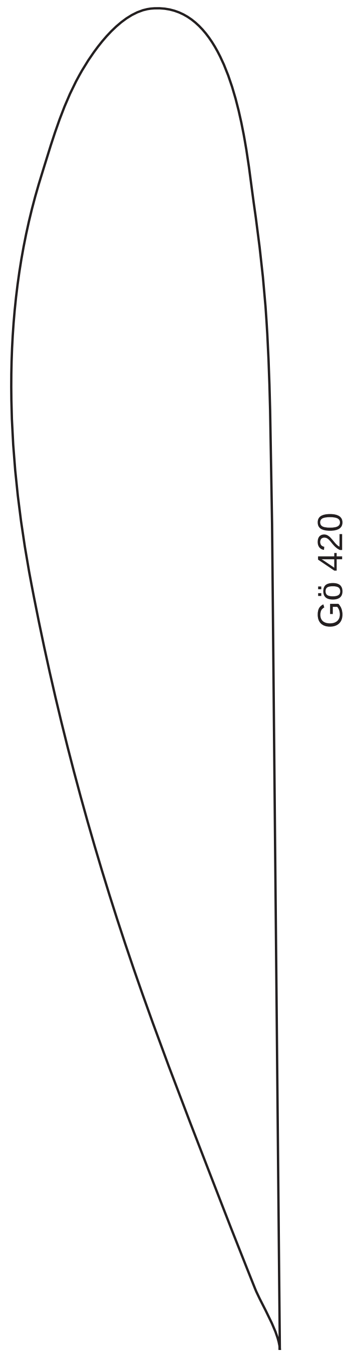
For the first cut at the problem we will make the following assumptions:

- The configuration will be that of the McDonnell-Douglas gyromill, using three wings.
- Wind turbine efficiency is 80%.
- The wings use the Göttingen-420 airfoil (see drawing at the end of the problem). We will accept the simple efficiency formula derived in this chapter.
- The wind turbine aspect ratio will be 0.8.
- The wings will be hollow aluminum blades. Their mass will be 25% of the mass of a solid aluminum wing with the same external shape. Incidentally, aluminum is not a good choice for wind turbine wings because it is subject to fatigue. Composites are better.
- The total wind turbine mass will be 3 times the mass of the three wings taken together.
- Estimated wind turbine cost is $1.00/kg. Notice that the cost of aluminum was $1200/kg in 1852, but the price is now down to $0.40/kg.
- The cost of investment is 12% per year.

Estimate:

a. The swept area.
b. The wing chord.
c. The wing mass.

Solution of Problem 15.7



Solution of Problem 15.7

d. The Reynolds number when the turbine is operating at its rated power. Assume that this occurs at the optimum U/V ratio.
e. The rpm at the optimum U/V ratio.
f. The torque under the above conditions.
g. The tension on each of the horizontal supporting beams (two beams per wing).
h. The investment cost per rated kW. Assuming no maintenance nor operating cost, what is the cost of the generated kWh? Use a utilization factor of 25%.

---

a. The swept area.

....

The knot is a unit of speed and corresponds to 1 nautical mile (1852 m) per hour. 14 knots correspond to

$$v = \frac{14 \times 1852}{3600} = 7.2 \quad \text{m/s}.$$

The power delivered by the turbine is

$$P_D = \frac{16}{27}\frac{1}{2}\rho v^3 \eta A_v = \frac{16}{27} \times \frac{1}{2} \times 1.29 \times 7.2^3 \times 0.8 A_v = 114 A_v = 10^6,$$

$$A_v = \frac{10^6}{114} = 8760 \quad \text{m}^2.$$

In this calculation, we used $\rho = 1.29$, i.e., we assumed STP conditions.

The swept area is 8760 m^2.

---

b. The wing chord.

....

The swept area, calculated above is equal to $2rH$,

$$2rH = 8760,$$

while the aspect ratio is

$$0.8 = \frac{H}{2r}$$

Eliminating H from the two preceding equations and solving for r and then for H, one gets

$$r = 52.3 \quad \text{m},$$

$$H = 83.7 \quad \text{m}.$$

Solution of Problem 15.7

From Equation 48 in the Text,

$$\eta = 7.39S,$$

$$S = \frac{0.8}{7.39} = 0.11.$$

But,

$$S = N\frac{K}{2r} = 3\frac{K}{2 \times 52.3} = 0.11,$$

$$K = 3.9 \quad \text{m.}$$

The chord is 3.9 m.

---

c. The wing mass.

A graphic analysis of the air foil shows that the cross-sectional area is about 0.143 times the chord squared. With $K = 3.9$ m, the cross-section is $A = 0.143 \times 3.9^2 = 2.18$ m^2.

If solid the volume of aluminum in each wing would be $V = 2.18 \times 83.7 = 182$ m^3 and the mass would be 182×2700=491,000 kg..

Since the wings are hollow, they mass only $0.25 \times 491 = 123$ tons. Three wings will mass 369 tons.

9C The wings mass 369 tons.

---

d. The Reynolds number when the turbine is operating at its rated power. Assume that this occurs at the optimum U/V ratio.

Optimum U/V for this airfoil is 6.5. Thus, $U = 6.5V = 6.5 \times 7.2 = 46.8$ m/s. The average linear velocity of the wing relative to the air is 46.8 m/s and the Reynolds number is

$$R = 70,000UK = 70,000 \times 46.8 \times 3.0 = 12.5 \times 10^6$$

The Reynolds number is 12.5 million.

---

e. The rpm at the optimum U/V ratio.

$$\omega = \frac{U}{r} = \frac{46.8}{52.3} = 0.895 \quad \text{rad/s},$$

Solution of Problem 15.7

$$rpm = \frac{\omega}{2\pi} \times 60 = 8.55.$$

The wind turbine rotates at 8.5 rpm.

---

f. The torque under the above conditions.

The torque is

$$\Upsilon = \frac{P_D}{\omega} = \frac{10^6}{0.895} = 1.12 \times 10^6 \quad \text{Newton meters.}$$

The torque is 1.1 Newton meters.

---

g. The tension on each of the horizontal supporting beams (two beams per wing).

The centrifugal force acting on each wing is

$$F_C = M\omega^2 r = 123,000 \times 0.895^2 \times 52.3 = 5.15 \times 10^6 \quad \text{N.}$$

Each beam has to absorb half of this force,

$$F_{beam} = \frac{5.15 \times 10^6}{2} = 2.58 \times 10^0 \quad \text{N.}$$

Each beam must absorb 2.6 MN or 263 tons in tension.

---

h. The investment cost per rated kW.

The total mass of the wind turbine is 3 times the aggregate mass of the wings; it is $3 \times 369 = 1107$ tons. At \$1.00/kg, this amounts to 1.1 million dollars of investment for the turbine rated at 1 MW. Per kW, the cost is \$1100.

The wind turbine costs \$1100 per kW rated power.

---

i. Assuming no maintenance nor operating cost, what is the cost of the generated kWh? Use a utilization factor of 25%.

Solution of Problem 15.7

Under the conditions of the problem and assuming that the land on which the turbine is installed is free, the yearly cost of capital is

$$C = 1.1 \times 10^6 \times 0.12 = 132,000 \quad \$/\text{year}.$$

If the turbine were to deliver full rated power all the time, it would generate 1000 kW ×8766 hours/year or 8.8 million kWh of electricity. However the utilization factor is 25% so the actual power delivered is 2.2 million kWh. It should be noted that the utilization factor is much higher than the usual 15 to 20% because the wind in the region is uncommonly steady.

The cost per kWh is

$$c = \frac{132 \times 10^3}{2.2 \times 10^6} = 0.06 \quad \$/\text{kWh}$$

The electricity would cos 6 cents per kWh.

Disregard the $h^{1/7}$ variation of wind speed with height. See the figure on the previous page with the outline of the Göttingen-420 airfoil.

Solution of Problem 15.7

Prob 15.8 A sailboat has a drag, F, given by

$$F = aW^2$$

where F is in newtons, W is the velocity of the boat with respect to the water (in m/s), and $a = 80$ kg/m.

The sail of the boat has a 10 m^2 area and a drag coefficient of 1.2 when sailing before the wind (i.e., with a tail-wind).

Wind speed is 40 km/h.

When sailing before the wind, what is the velocity of the boat? How much power does the wind transfer to the boat? What fraction of the available wind power is abstracted by the sail?

....

Wind velocity is $V = 40$ km/h or 11.1 m/s.

In steady state, the drag force must equal the propulsive force, F_p.

$$F_p = \underbrace{\tfrac{1}{2}\rho(V-W)^2 C_D A}_{\text{Force of wind}} = \underbrace{aW^2}_{\text{Boat drag}}$$

$$(V-W)^2 - \frac{2a}{\rho C_D A}W^2 = 0$$

Let $\beta \equiv \frac{2a}{\rho C_D A} = \frac{2\times 80}{1.29\times 1.2\times 10} = 10.3$, then

$$(V-W)^2 - \beta W^2 = 0$$

from which

$$W = V\frac{1-\sqrt{\beta}}{1-\beta}.$$

Since $\beta > 1$, only the "$-$" in front of $\sqrt{\beta}$ can be used.

$$W = 11.1 \times 0.238 = 2.63 \text{ m/s} = 9.49 \text{ km/h} = 5.12 \text{ knots}.$$

Boat speed when sailing before the wind is 5.12 knots.

The power used by the boat is the product of the force on it times the resulting velocity:

$$P_B = aW^3 = 80 \times 2.63^3 = 1470\ W.$$

Power to propel the boat is 1470 W.

The available wind power is

$$P_A = \frac{16}{27} \times \tfrac{1}{2}\rho V^3 A = \frac{16}{27} \times \tfrac{1}{2} \times 1,29 \times 11.1^3 \times 10 = 5240 \text{ W}.$$

This means that the boat uses from the available wind power a fraction

$$\frac{P_B}{P_A} = \frac{1470}{5240} = 0.280.$$

The sail uses 28% of the available wind power.

Solution of Problem 15.8

Prob 15.9 A vertical-axis windmill of the gyromill configuration extracts (as useful power) 50% of the available wind power. The windmill has a rectangular swept area with a height, H, of 100 m and an aspect ratio of 0.8.

The lower tips of the wings are 10 m above ground. At this height, the wind velocity is 15 m/s. It is known that the wind increases in velocity with height according to the 1/7th power law.

Assuming that the windmill is at sea level, what power does it generate?

....

The power developed per area increment (horizontal slices) is

$$dP = \frac{16}{27} \times \tfrac{1}{2}\rho V^3 dA,$$

where $V(h) = V\left(\frac{h}{h_0}^{1/7}\right)$ and $dA = 2rdh$.

$$C \equiv \frac{16}{27} \times \tfrac{1}{2}\rho\eta = \frac{16}{27} \times \tfrac{1}{2} \times 1.29 \times 0.5 = 0.191,$$

$$P = C\int_{h_0}^{h_0+H} V^3 dA = Ch_0^{-3/7} V_0^3 \times 2r \int_{h_0}^{h_0+H} h^{3/7} dh,$$

$$P = 0.19 \times 10^{-3/7} \times 15^3 \times 2r \int_{h_0}^{h_0+H} h^{3/7} dh = 481r \int_{h_0}^{h_0+H} h^{3/7} dh,$$

$$\Lambda = \frac{H}{2r} \qquad \therefore \qquad r = \frac{H}{2\Lambda} = \frac{100}{2 \times 0.8} = 62.5 \qquad \text{m},$$

$$P = 30,000 \int_{h_0}^{h_0+H} h^{3/7} dh = 30,000 \times \frac{7}{10}\left[(h_0 + H)^{10/7} - h_0^{10/7}\right]$$

$$= 21,000\left[(10 + 100)^{10/7} - 10^{10/7}\right]$$

$$= 16,750,000 \quad \text{W} \quad \text{or} \quad 16.75 \text{ MW}.$$

The power generated by the windmill is 16.75 MW.

Solution of Problem 15.9

Prob 15.10 *Solve equations by trial and error. Use a computer*

Can a wind-driven boat sail directly into the wind? Let's find out (forgetting second order effects).

As a boat moves through the water with a velocity, W (relative to the water), a drag force, F_w, is developed. Let $F_w = 10W^2$.

The boat is equipped with a windmill having a swept area of 100 m^2 and an overall efficiency of 50%. The power generated by the windmill is used to drive a propeller which operates with 80% efficiency and creates a propulsive force, F_p, that drives the boat.

The windmill is oriented so that it always faces the *induced* **wind, i.e., the combination of V and W.**

The wind exerts a force, F_{WM}, on the windmill. This force can be estimated by assuming a $C_D = 1.1$ and taking the swept area as the effective area facing the wind.

Wind velocity, V, is 10 m/s. What is the velocity, W_S, of the boat in the water? Plot W as a function of the angle, ϕ, between the direction of the wind and that of the boat. In a tail wind, $\phi = 0$ and in a head wind, $\phi = 180°$.

The boat has a large keel, so that the sideways drift caused by the "sail" effect of the windmill is negligible.

....

The power generated by the windmill is

$$P_g = \frac{16}{27}\frac{1}{2}\rho\eta A U^3 = \frac{16}{27}\frac{1}{2}1.29 \times 0.5 \times 100\ U^3 = 19.1\ U^3. \qquad (1)$$

where U is the induced wind velocity.

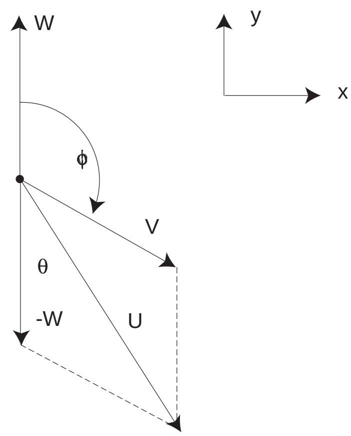
The force developed by the boat propeller is

$$F_p = \eta_p \frac{P_g}{W} = 15.29\frac{U^3}{W}. \qquad (2)$$

The above equation only makes sense over a limited range of W. If W is zero (if the boat is anchored or moored), the force predicted above would be infinite. The efficiency η_p becomes zero under such conditions.

Let ϕ be the direction of the wind relative to the direction of the boat motion. In a tail wind, $\phi = 0$ and in a head wind, $\phi = 180°$.

Solution of Problem 15.10



The induced wind is

$$\vec{U} = -\vec{W} + \vec{V}. \tag{3}$$

Using an appropriate coordinate system,

$$\vec{W} = \vec{j}W, \tag{4}$$

$$\vec{V} = \vec{i}V\sin\phi + \vec{j}V\cos\phi, \tag{5}$$

$$\vec{U} = \vec{i}V\sin\phi + \vec{j}(V\cos\phi - W), \tag{6}$$

$$U = \sqrt{V^2 + W^2 - 2VW\cos\phi}. \tag{7}$$

The drag on the windmill is along the $\vec{U}$ direction and has a magnitude

$$F_{WM} = \frac{1}{2}\rho C_D A U^2 = \frac{1}{2}\ 1.29 \times 1.1 \times 100\ U^2 = 70.95\ U^2. \tag{8}$$

The component of drag parallel to the direction of motion of the boat is

$$F_{W_{WM}} = F_{WM}\cos\theta, \tag{9}$$

where θ is the angle between $-W$ and U. Don't confuse this angle with the "wind angle", ϕ, defined previously.

This angle can be found by noticing that

$$\vec{U}\cdot(-\vec{W}) = -UW\cos\theta. \tag{10}$$

From this,

$$\cos\theta = \frac{W^2 - VW\cos\phi}{UW} = \frac{W - V\cos\phi}{\sqrt{V^2 + W^2 - 2VW\cos\theta}}. \tag{11}$$

Solution of Problem 15.10

and

$$F_{W_{WM}} = 70.95(W - V\cos\phi)\sqrt{V^2 + W^2 - 2VW\cos\phi} \tag{12}$$

The water drag on the boat is

$$F_W = 10\ W^2. \tag{13}$$

Under all circumstances, the force, F_p, generated by the propeller must balance the wind drag, $F_{W_{WM}}$, plus the water drag, F_W:

$$F_p - F_W - F_{W_{WM}} = 0. \tag{14}$$

$$15.29\frac{U^3}{W} - 10\ W^2 - 70.95(W - V\cos\phi)U = 0. \tag{15}$$

The solution to the problem is found by

– taking selected values of ϕ (say, every 10°),
– setting $W = 10$ m/s (an arbitrary value),
– calculating U from Equation 7,
– finding by trial an error the value of W that satisfies Equation 15.

See plot of W as a function of ϕ.

One can check the results in a simple manner at $\phi = 0$ and $\phi = 180°$.

In both cases, $V = 10$ m/s, $F_p = 15.29\frac{U^3}{W}$, $F_{W_{WM}} = 70.95U^2$ and $F_W = 10W^2$.

For $\phi = 0$, $U = V - W$ and $F_{W_{WM}} + F_p = F_W$. From this,

$$10W^2 - \frac{15.29(10 - W)^3}{W} - 70.95(10 - W)^2 = 0$$

The root of the above equation is $W = 7.34$ m/s.

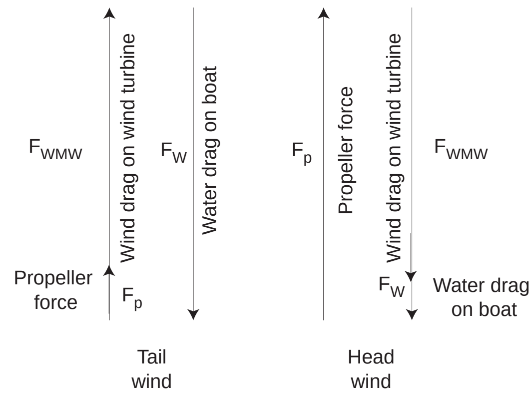
For $\phi = 180°$, $U = V + W$ and $F_p = F_{W_{WM}} + F_W$. From this,

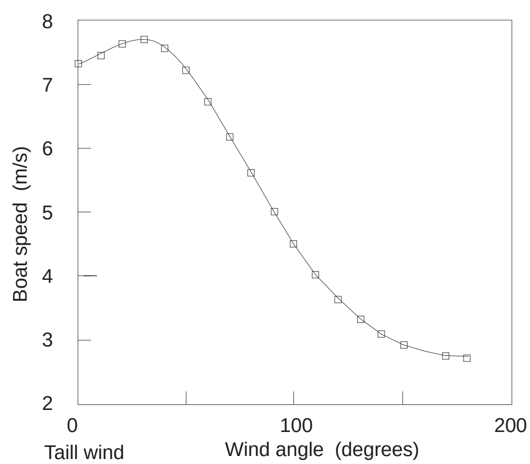
$$10W^2 - \frac{15.29(10 + W)^3}{W} + 70.95(10 + W)^2 = 0$$

The root is $W = 2.72$ m/s.

These values of W are the same found in the general solution.

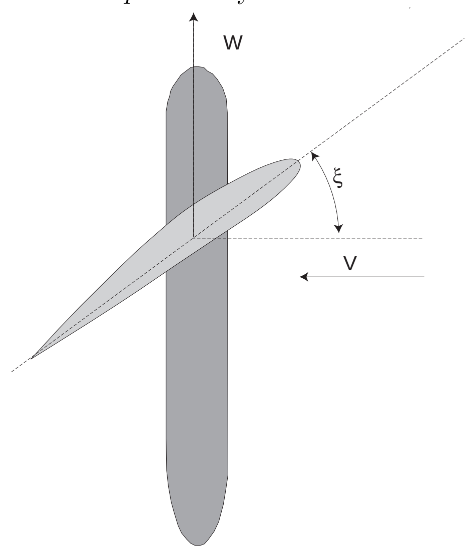
Solution of Problem 15.10





Solution of Problem 15.10

Prob 15.11 *Solve equations by trial and error. Use a computer*



A vehicle is mounted on a rail so that it can move in a single direction only. The motion is opposed by a drag force, F_W,

$$F_W = 100W + 10W^2$$

where W is the velocity of the vehicle along the rail. Notice that the drag force above does not include any aerodynamic effect of the "sail" that propels the vehicle. Any drag on the sail has to be considered in addition to the vehicle drag.

The wind that propels the vehicle is perpendicular to $\vec{W}$, and has a velocity, $V = 10$ m/s. It comes from the starboard side (the right side of the vehicle).

The "sail" is actually an airfoil with an area of 10.34 m^2. It is mounted vertically, i.e., its chord is horizontal and its length is vertical. The reference line of the airfoil makes an angle, ξ, with the normal to $\vec{W}$. See the figure.

The airfoil has the following characteristics:

$$C_L = 0.15\alpha,$$

$$C_D = 0.015 + 0.015|\alpha|.$$

α is the angle of attack expressed in degrees. The two formulas above are valid only for $-15° < \alpha < 15°$. If α exceeds 15°, the wing stalls and the lift falls to essentially zero.

Solution of Problem 15.11

Conditions are STP.

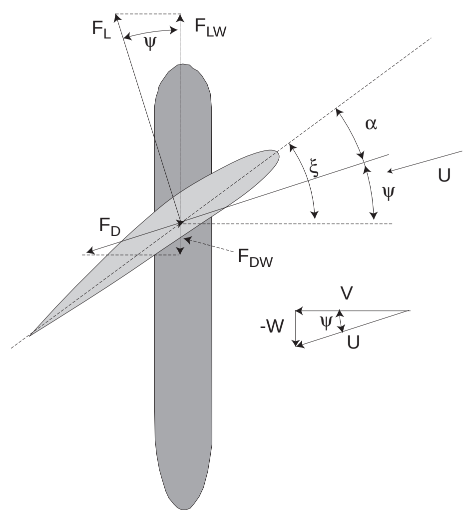
Calculate the velocity, W, with which the vehicle moves (after attaining a steady velocity) as a function of the setup angle, ξ. Plot both W and α as a function of ξ.

....

Please refer to the figure .

The action of the wind on the air foil generates a lift force, F_L, whose component along the direction of the motion of the vehicle, i.e., the direction of $\vec{W}$, is F_{LW}. This is the only propulsive force in the problem. In steady state, it must be exactly balanced by the sum of the drag forces. These are F_W the drag on the vehicle itself and the aerodynamic drag, F_D, projected along the direction of motion of the vehicle. This is F_{DW}. Hence,

$$F_{LW} = F_W + F_{DW}. \tag{1}$$



The lift force on the air foil is

$$F_L = \frac{1}{2}\rho A C_L U^2 = \frac{1}{2}\ 1.29 \times 10.34 \times 0.15\alpha U^2 = \alpha U^2 \tag{2}$$

From the vector sum,

$$\vec{U} = \vec{V} - \vec{W}, \tag{3}$$

where $\vec{U}$ is the induced wind velocity, $\vec{W}$ is the vehicle velocity and $\vec{V}$ is the velocity of the wind, we have,

$$U^2 = V^2 + W^2. \tag{4}$$

Hence,

$$F_L = \alpha(V^2 + W^2) \tag{5}$$

Solution of Problem 15.11

and

$$F_{LW} = F_L \cos\psi = \alpha(V^2 + W^2)\cos\psi. \tag{6}$$

Still from Equation 3,

$$\psi = \arctan\frac{W}{V}. \tag{7}$$

$$F_R = \frac{1}{2}\rho A U^2(0.015 + 0.015|\alpha|) = 0.1(1 + |\alpha|)(V^2 + W^2) \tag{8}$$

and

$$F_{DW} = 0.1(1 + |\alpha|)(V^2 + W^2)\sin\psi. \tag{9}$$

From the problem statement,

$$F_W = 100W + 10W^2. \tag{10}$$

Introducing Equations 6, 9 and 10 into Equation 1,

$$\alpha(V^2 + W^2)\cos\psi - 0.1(1 + |\alpha|)(V^2 + W^2)\sin\psi$$
$$- 100W - 10W^2 = 0 \tag{11}$$

From the figure, we see that

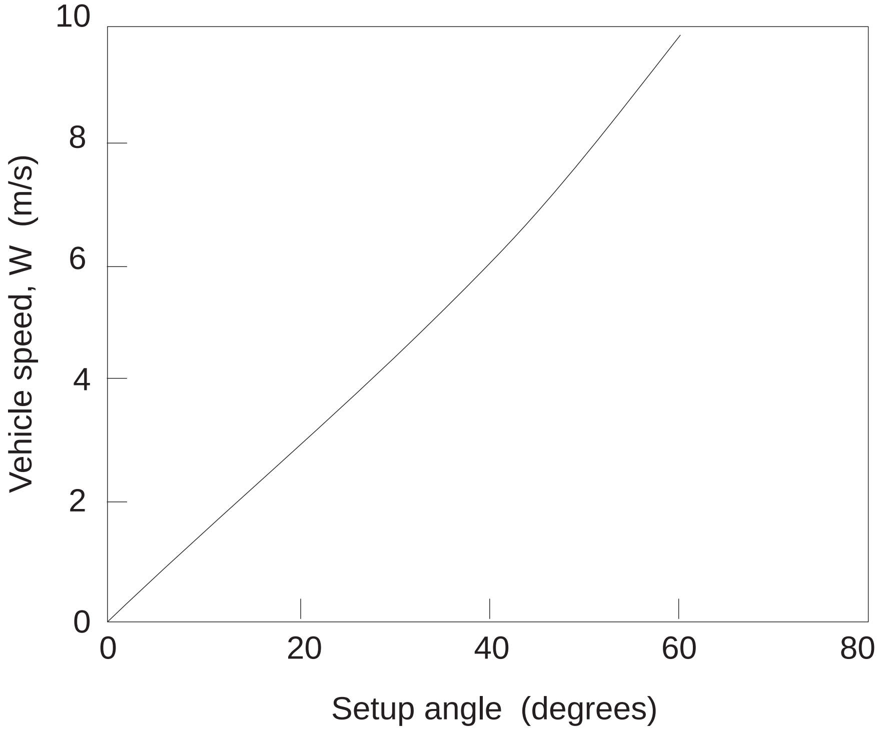
$$\alpha = \xi - \psi. \tag{12}$$

To solve the problem, we create a spread-sheet as shown below, using Excel. The first column is a list of arbitrarily selected setup angles, ξ. Column 2 is the vehicle velocity (see later). From Equation 7, the corresponding value of ψ is calculated in Column 3, and, from Equation 12, the value of α in Column 4. Finally, the left-hand-side of Equation 11 is calculated in Column 5. If the value of W in Column 2 had been entered correctly, Column 5 would add up to zero. The solution is found by successive guess of the value of W entered manually until the value in Column 5 is sufficiently small.

Setup angle ξ	Vehicle velocity W	ψ	Attack angle α	Residue
0	0	0	0	0.000
5	0.7345	4.201	0.799	-0.038
10	1.4598	8.305	1.695	-0.007
15	2.184	12.32	2.68	-0.003
20	2.9145	16.25	3.751	-0.086
25	3.6579	20.09	4.908	0.000
30	4.4214	23.85	6.148	-0.006
35	5.2118	27.53	7.472	0.032
40	6.0363	31.12	8.884	-0.031
45	6.9019	34.61	10.39	0.028
50	7.8165	38.01	11.99	-0.026
55	8.7875	41.31	13.69	-0.014
60	9.8226	44.49	15.51	0.010

A plot of W versus ξ is shown in the figure that appears in the next page.

Solution of Problem 15.11



Solution of Problem 15.11

Prob 15.12 An electric generator, rated at 360 kW at 300 rpm and having 98.7% efficiency at any reasonable speed, produces power proportionally to the square of the number of rpm with which it is driven.

This generator is driven by a windmill that, under given wind conditions, has a torque of 18,000 Nm at 200 rpm, but produces no torque at both 20 and 300 rpm. Assume that the torque varies linearly with the number of rpm between 20 and 200 and between 200 and 300 rpm.

What is the electric power generated?

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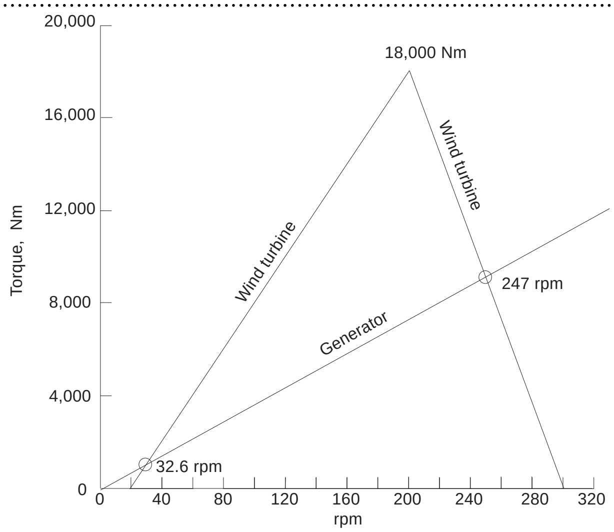


20 rpm; $\omega = 2.09$ rad/s.
200 rpm; $\omega = 20.95$ rad/s.
300 rpm; $\omega = 31.42$ rad/s.

Since the generator is 98.7% efficient, it takes $360/0.987 = 365$ kW to drive it at 300 rpm (31.42 rad/s):

$$P = a\omega^2 = a \times 31.42^2 2 = 365,000. \tag{1}$$

$$P_g = 370\omega^2 \quad \text{W}. \tag{2}$$

Consequently, the torque is

$$\Upsilon = 370\omega \quad \text{Nm}. \tag{3}$$

According to the statement of the problem, the torque of the wind turbine is linearly related to its angular velocity

$$\Upsilon_{WT} = a_b\omega : \tag{4}$$

Solution of Problem 15.12

In the range between 20 and 200 rpm:
$\omega = 2.09$ rad/s, $\Upsilon_{WT} = 0$,
$\omega = 20.95$ rad/s, $\Upsilon_{WT} = 18,000$ Nm.
From this,

$$\Upsilon_{WT} = -1,995 + 954.4\omega. \tag{5}$$

In the range between 200 and 300 rpm:
$\omega = 20.95$ rad/s, $\Upsilon_{WT} = 18,000$,
$\omega = 31.42$ rad/s, $\Upsilon_{WT} = 0$.
From this,

$$\Upsilon_{WT} = 54,017 - 1719\omega. \tag{6}$$

At equilibrium, $\Upsilon_G = \Upsilon_{WT}$. This leads to two solutions:

$$\omega = 3.41 \quad \text{rad/s} \quad \text{or} \quad 32.6 \quad \text{rpm} \tag{7}$$

and

$$\omega = 25.86 \quad \text{rad/s} \quad \text{or} \quad 247 \text{ rpm}. \tag{8}$$

The lower rpm solution is unstable: if a gust of wind accelerates the wind turbine, its torque will increase and, becoming larger than that of the generator, will continue increasing until the stable, higher rpm, solution is reached. At this point, the generator will absorb

$$P_g = 370 \times 25.86^2 = 247,400 \quad \text{W}, \tag{9}$$

and will deliver

$$P_D = 0.987 \times 247.4 = 244 \quad \text{kW}. \tag{10}$$

The generator will deliver 244 kW at 247 rpm.

Solution of Problem 15.12

Prob 15.13 A car is equipped with an electric motor capable of delivering a maximum of 10 kW of mechanical power to its wheels. It is on a horizontal surface. The rolling friction (owing mostly to tire deformation) is 50 N regardless of speed. There is no drag component proportional to the velocity. Frontal area is 2 m^2 and the aerodynamic drag coefficient is $C_D = 0.3$.

Assume calm air at 300 K and at 1 atmosphere. What is the cruising speed of the car at full power?

With the motor uncoupled and a tailwind of 70 km/h, what is the car's steady state velocity. The drag coefficient is $C_D = 1$ when the wind blows from behind.

The aerodynamic force acting on the car is

$$F_W = \frac{1}{2}\rho V^2 C_D A,$$

where V is the air velocity relative to the car.

At 1 atmosphere pressure and 300 K, the air density is

$$\rho = \frac{(0.2 \times 32 + 0.8 \times 28) \times 1.013 \times 10^5}{8314 \times 300} = 1.17 \quad \text{kg/m}^3.$$

Let F_D be the rolling friction and P the power to move the car. Then,

$$P = (F_W + F_D)V = \frac{1}{2}\rho V^3 C_D A + 50V = \frac{1}{2}\ 1.17 \times 0.3 \times 2V^3 + 50V$$
$$= 0.353V^3 + 50V = \underbrace{10^4}_{motorpower} \quad \text{W}$$

By trial and error, the solution, $V = 28.97$ m/s, is found.

The car cruises at 28.97 m/s or 104.3 km/h.

When there is no power from the motor and the wind (70 km/h or 19.4 m/s) blows from behind, The force of the wind, F_W, must balance the drag of the car, F_D. The wind relative to the car is $V - U = 19.4 - U$ where U is the car velocity.

$$\frac{1}{2}\rho V^2 C_D A = 50,$$

$$\frac{1}{2}1.17 \times 1 \times 2 \times (19.4 - U)^2 = 50$$

$$(19.4 - U)^2 = 42.7.$$

The solution is 12.86 m/s or 46.3 km/h.

The wind will push the car at a speed of 46.3 km/h.

Solution of Problem 15.13

Prob 15.14 A building is 300 m tall and 50 m wide. Its C_D is 1. What is the force that the wind exerts on it if it blows at a speed of 10 m/s at a height of 5 m and if its velocity varies with $h^{1/7}$ (h being height above ground).

The pressure on the building is

$$p = \frac{1}{2}\rho V^2 C_D = 0.645 V^2 \qquad \text{N m}^{-2}.$$

Here, we assumed sea level ($\rho = 1.29$ kg m^{-3}).
The wind velocity scales with height as

$$V(h) = V(h_0)\left(\frac{h}{h_0}\right)^{1/7} = 10\left(\frac{h}{5}\right)^{1/7},$$

thus, the pressure is

$$p = 0.645 \times 10^2 \left(\frac{h}{5}\right)^{2/7} = 63.1 h^{2/7}.$$

On each elementary horizontal slice on the building's face, the force is

$$dF = pdA = 63.1 h^{2/7} \times 50 dh = 3157 h^{2/7} dh,$$

and the total force is

$$F = 3157 \int_0^{300} 3157 h^{2/7} dh = 2455 h^{9/7}\Big|_0^{300} = 3.76 \times 10^6 \qquad \text{N}.$$

The force on the building is 3.76 MN or 383 tons.

Solution of Problem 15.14

Prob 15.15 A windmill has a torque versus angular velocity characteristic that can be described by two straight lines passing through the points:

50 rpm, torque = 0;
100 rpm, torque = 1200 Nm;
300 rpm, torque = 0.

1. **If the load absorbs power according to $P_{load} = 1000\omega$, what is the power taken up by this load? What is the torque of the windmill? What is the angular velocity , ω?**
2. **If you can adjust the torque characteristics of the load at will, what is the maximum power that you can extract from the windmill? What is the corresponding angular velocity?**

....

According to the problem statement, the torque is given by

$$\Upsilon_W = a\omega + b.$$

Let us fit two segments of the torque versus angular velocity characteristic:

At 50 rpm, $\omega = \omega_1 = (5.24 \text{ rad/s})$,
At 100 rpm, $\omega = \omega_2 = (10.47 \text{ rad/s})$,
At 300 rpm, $\omega = \omega_3 = (31.42 \text{ rad/s})$.
The coefficients, a and b can be calculated:
For $\Upsilon_W \leq 1200$ N m,

$$\Upsilon_W = 229.4\omega - 1202.$$

For $\Upsilon_W \geq 1200$ N m,

$$\Upsilon_W = -57.28\omega + 1800.$$

Since

$$P_{LOAD} = 1000\omega.$$

it follows that

$$\Upsilon_{LOAD} = 1000 \quad \text{N m}.$$

In the region $\Upsilon_W \leq 1200$,

$$1000 = 229.4\omega - 1202,$$

$$\omega = 9.59 \quad \text{rad/s}.$$

Solution of Problem 15.15

However, this is an unstable point. See Subsection 2.12 of the text book.

In the region $\Upsilon_W \geq 1200$,

$$1000 = -57.28\omega + 1800,$$

$$\omega = 13.97 \quad \text{rad/s},$$

$$P = 1000 \times 13.97 = 14.000 \quad \text{W}.$$

The windmill generates 14 kW. The torque is 1000 N m. The angular velocity is 13.97 rad/s or 133 rpm.

Clearly, the power that can be extracted from a windmill depends on the matching of the windmill characteristics to that of the generator. The maximum power occurs when $P = \Upsilon\omega = -57.28\omega^2 + 1800\omega$ is maximum.

$$\frac{dP}{d\omega} = -114.6\omega + 1800 = 0.$$

A maximum power of 14.1 kW can be extracted from the windmill when it operates at 15.7 rad/s or 150 rpm.

Solution of Problem 15.15

Prob 15.16 A wind turbine with a swept area of 1000 m^2 operates with 56% efficiency under STP conditions.

At the location of the windmill, there is no wind between 1800 and 0600. At 0600, the wind starts up and its velocity increases linearly with time from zero to a value that causes the 24-h average velocity to be 20 m/s. At 1800, the wind stops abruptly.

What is the maximum energy the windmill can generate in one year?

....

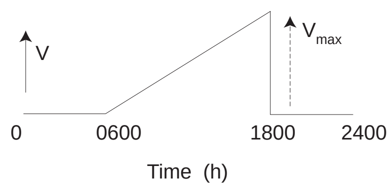
The power generated by the windmill is

$$P_g = \frac{16}{27} \times \tfrac{1}{2}\rho V^3 A_v \eta = \frac{16}{27} \times \tfrac{1}{2} \times 1.29 \times 1000 \times 0.65 V^3 = 248 V^3$$

The *average* power generated is

$$<P_g>= 248 <V^3> .$$

We must find the average value of V^3.
The wind velocity behaves as shown below:



The average or mean value of V is

$$<V> \frac{V_{max} \times 12}{2 \times 24} = \frac{V_{max}}{4} = 20 \quad \text{m/s}.$$

$$v_{max} = 80 \quad \text{m/s}.$$

For $0 < t < 6$ and $1800 < t < 2400$, $v = 0$;
For $6 < t < 18$, $v = 6.67t - 40$.
Shift the origin of time ($t' = t - 6$),

$$<V^3> = \frac{1}{24}\int_0^{12} (6.67\, t')^3 dt' = \frac{1}{24}\int_0^{12} 296.3 t'^3 dt'$$

$$= \frac{296.3}{24} \frac{t'^4}{4}\bigg|_0^{12} = 64,000 \quad (\text{m/s})^3$$

$$<P_g>= 248 \times 64,000 = 15.87 \quad \text{MW}$$

One year contains 31.6×10^6 seconds. Thus,

$$W = 31.6 \times 10^6 \times 15.87 \times 10^6 = 500 \times 10^{12} \quad \text{J}.$$

The windmill generates 500 TJ in one year.

Solution of Problem 15.16

Prob 15.17 **U.S. Windpower operates the Altamont Wind Farm near Livermore, CA. They report an utilization factor of 15% for their 1990 operation. Utilization factor is the ratio of the energy generated in one year, compared with the maximum a plant would generate if operated constantly at the rated power. Thus, a wind turbine rated at 1 kW would produce an average of 150 W throughout the year. Considering the intermittent nature of the wind, a 15% utilization factor is good. Hydroelectric plants tend to operate with a 50% factor.**

1. **It is hoped that the cost of the kWh of electricity will be as low as 5 cents. Assuming that the operating costs per year amount to 15% of the total cost of the wind turbine and that the company has to repay the bank at a yearly rate of 12% of the capital borrowed, what cost of the wind turbine cannot be exceeded if the operation is to break even?**

   For your information, the cost of a fossil fuel or a hydroelectric plant runs at about 1000 dollars per installed kW.
2. **If, however, the wind turbine actually costs \$1000 per installed kW, then, to break even, what is the yearly rate of repayment to the bank?**

---

1. **It is hoped that the cost of the kWh of electricity will be as low as 5 cents. Assuming that the operating costs per year amount to 15% of the total cost of the wind turbine and that the company has to repay the bank at a yearly rate of 12% of the capital borrowed, what cost of the wind turbine cannot be exceeded if the operation is to break even?**

   For your information, the cost of a fossil fuel or a hydroelectric plant runs at about 1000 dollars per installed kW.

....

The average power of the wind turbine is 150 W per installed kW. Hence, per year and per installed kW, the wind turbines will generate $150 \times 3.16 \times 10^7 = 4.7 \times 10^9$ joules or $4.7 \times 10^9/3.6 \times 10^6 = 1317$ kWh. Here 3.16×10^7 is the approximate number of seconds in a year and 3.6×10^6 is the factor that converts joules into kWh.

At a price of \$0.05 per kWh, the energy sold will yield $1317 \times 0.05 =$ \$65.8 per year per installed kW.

If C is the total cost of the wind turbine, then the annual cost of operation (per installed kW) is

$$\underbrace{0.15C}_{\text{operation cost}} + \underbrace{0.12C}_{\text{investment cost}} = 0.27C$$

Solution of Problem 15.17

$$C = \frac{65.8}{0.27} = \$244.$$

The cost of the wind turbine cannot exceed $244/kW.

As stated, present day cost of wind turbines is about $1000/kW. So, it appears that to sell electricity at $0.05/kWh, both the operation and the investment cost must be substantially lower.

---

2. If, however, the wind turbine actually costs $1000 per installed kW, then, to break even, what is the yearly rate of repayment to the bank?

....

Let x be the sum of the fractional cost of operation plus investment. This, multiplied by the total wind turbine cost, must equal the annual revenue (all per installed kW):

$$1000x = 62.5,$$

$$x = 0.0625.$$

The sum of operation plus investment cost cannot exceed 6.25%. If the operation cost is still 15%, then the produced electric energy will cost much more than $0.05/kWh.

It seems implausible that the wind farm can produce electricity as cheaply as announced. This is one reason for "green pricing", an arrangement by which customers agree to pay a higher price for energy generated from renewable sources.

Note added in 2009:

Present day cost of wind turbines is about 1,200 $/kW. However utilization factors of nearly 30% (in stead of the 15% used in this problem) have been achieved. In addition the price of electricity is more like 0.15 $/kWh, not 0.05 $/kW. So, it appears wind generated electricity can now be profitably sold.

Solution of Problem 15.17

Prob 15.18 Many swimmers specialize in both free style (crawl) and butterfly. The two strokes use almost the same arm motion, but the crawl uses an alternate stroke whereas the butterfly uses a simultaneous one. The kicks are different but are, essentially, equally effective. Invariably, swimmers go faster using the crawls.

From information obtained in this course, give a first order explanation of why this is so.

....

The major difference is in that, owing to the alternate stroke, the crawl swimmer maintains a more constant speed while the speed of the butterfly swimmer, varies substantially during the stroke. Since the water drag is proportional to the square of the speed, it is easy to see that, given the same average velocity, the drag is larger in the fly than in the crawl.

Solution of Problem 15.18

Prob 15.19 *Solve equations by trial and error. Use a computer*

An air foil of uniform chord is mounted vertically on a rail so that it can move in a single direction only. It is as if you had cut off a wing of an airplane and stood it up with its longer dimension in the vertical. The situation is similar to the one depicted in the figure of Problem 15.11.

There is no friction in the motion of the air foil. The only drag on the system is that produced by the drag coefficient of the airfoil.

The *set-up angle* **is the angle between the airfoil reference plane and the normal to the direction of motion. In other words, if the rail is north-south, the air foil faces east when the set-up angle is zero and faces north when the set-up angle is 90°.**

Consider the case when the set-up angle is zero and the wind blows from the east (wind is abeam). Since the wind generates a lift, the airfoil will move forward. What speed does it attain?

Conditions are STP. Area of the airfoil is 10 m^2. Wind speed is 10 m s^{-1}

The coefficients of lift and drag are given by

$$C_L = 0.6 + 0.066\alpha - 0.001\alpha^2 - 3.8 \times 10^{-5}\alpha^3.$$

$$C_D = 1.2 - 1.1\cos\alpha.$$

....

The angle, ψ, is given by

$$\tan\psi = \frac{W}{V}.$$

When the setup angle, ζ, is zero, $\alpha = -\psi$.

The lift force, F_L, is

$$F_L = \tfrac{1}{2}\rho U^2 A C_L$$

and the drag force, F_D, is

$$F_D = \tfrac{1}{2}\rho U^2 A C_D$$

where U is the induced wind velocity and A is the area of the airfoil.

The component of these forces along the direction of motion are

$$F_{LB} = \tfrac{1}{2}\rho U^2 A C_L \cos\psi$$

and

$$F_{DB} = \tfrac{1}{2}\rho U^2 A C_D \sin\psi.$$

A steady velocity is reached when $F_{LB} = F_{DB}$ or

$$C_L \cos\psi = C_D \sin\psi$$

Solution of Problem 15.19

or

$$\frac{C_L}{C_D} = \tan\psi = -\tan\alpha. \tag{1}$$

This leads to

$$\frac{0.6 + 0.066\alpha - 0.001\alpha^2 - 3.8 \times 10^{-5}\alpha^3}{1.2 - 1.1\cos\alpha} + \tan\alpha = 0.$$

This is a transcendental equation; it can be solved numerically. The result is $\alpha = -8.15°$. Consequently, $\psi = 8.15°$ and

$$W = V\tan\psi = 10\tan 8.15° = 1.43 \quad \text{m/s}.$$

The airfoil speed is 1.43 m/s.

Solution of Problem 15.19

Prob 15.20 Consider a vertical rectangular (empty) area with an aspect ratio of 0.5 (taller than wide) facing a steady wind. The lower boundary of this area is 20 m above ground and the higher is 200 mm above ground. The wind velocity at 10 m is 20 m/s and it varies with height according to the $h^{1/7}$ law.

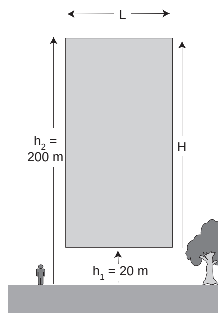
Calculate:

1. **the (linear) average velocity of the wind over this area,**
2. **the cubic mean velocity of the wind over this area,**
3. **the available wind power density over this area,**
4. **the mean dynamic pressure over this area.**

Assume now that the area is solid and has a C_D of 1.5. Calculate:

5. **the mean pressure over this area,**
6. **the torque exerted on the root of a vertical mast on which the area is mounted.**

....



The wind speed obeys

$$V(h) = 20\left(\frac{h}{h_0}\right)^{1/7}.$$

The various averages are

$$<V>_i = \left[\frac{1}{H}\int_{h_1}^{h_2} V^i dh\right]^{1/i} = \left[\frac{1}{H}\int_{h_1}^{h_2} \frac{20^i}{h_0^{i/7}} h^{i/7} dh\right]^{1/i}$$

$$= \left[\frac{1}{H}\left(\frac{20}{10^{1/7}}\right)^i \int_{h_1}^{h_2} h^{i/7} dh\right]^{1/i}$$

$$= \left[\frac{1}{180}\left(\frac{20}{10^{1/7}}\right)^i \frac{7}{i+7}\left(h_2^{\frac{i+7}{7}} - h_1^{\frac{i+7}{7}}\right)\right]^{1/i}$$

Solution of Problem 15.20

The results are:

i		$<V>_i$
1	Linear mean	27.68 m/s
2	Quadratic mean	27.77 m/s
3	Cubic mean	27.86 m/s

In this particular example, there is but a negligible difference between the three means.

The cubic mean velocity of the wind over the area is 27.86 m/s.

The available power is

$$P_A = \frac{16}{27} \times \tfrac{1}{2}\rho <V>_3^3 \quad \mathrm{W/m^2},$$

$$P_A = \frac{16}{27} \times \tfrac{1}{2} 1.29 \times 27.86^3 = 8265 \mathrm{W/m^2}.$$

The cubic mean velocity of the wind over the area is 27.86 m/s.

The dynamic pressure, averaged over the area is

$$p_{dyn} = \tfrac{1}{2}\rho <V>_2^2 = \tfrac{1}{2} \times 1.29 \times 27.77^2 = 497 \quad \mathrm{N/m^2}.$$

The mean dynamic pressure on the area is 497 $\mathrm{N/m^2}$.

The mean actual pressure of the area is

$$p_{mean} = C_D p_{dyn} = 1.5 \times 497 = 796 \quad \mathrm{N/m^2}.$$

The mean actual pressure on the area is 746 $\mathrm{N/m^2}$.

Each elementary horizontal slice of the area contributes the elementary torque

$$\begin{aligned} d\Upsilon &= h dF = h \times \tfrac{1}{2}\rho C_D V^2 L dh \\ &= \tfrac{1}{2} \times 1.29 \times 1.5 \times 20^2 \times \frac{1}{10^{2/7}} h^{9/7} dh = 18,040 h^{9/7} dh, \end{aligned}$$

$$\begin{aligned} \Upsilon &= 18,040 \int_{h_1}^{h_2} h^{9/7} dh = 18,040 \frac{7}{16}\left(h_2^{16/7} - h_1^{16/7}\right) \\ &= 7890\left(200^{16/7} - 20^{16/7}\right) = 1.43 \times 10^9 \quad \mathrm{N\ m}. \end{aligned}$$

The wind exerts a torque of 1.43 GN m at the root of the mast.

Solution of Problem 15.20

Prob 15.21 The retarding force, F_D, on a car can be represented by

$$F_D = a_0 + a_1 V + a_2 V^2.$$

To simplify the math, assume that $a_1 = 0$.

A new electric car is being tested by driving it on a perfectly horizontal road on a windless day. The test consists of driving the vehicle at constant speed and measuring the energy used up from the battery. Exactly 15 kWh of energy is used in each case.

When the car is driven at a constant 100 km/h, the distance covered is 200 km. When the speed is reduced to 60 km/h, the distance is 362.5 km.

If the effective frontal area of the car is 2.0 m^2, what is the coefficient of aerodynamic drag of the vehicle?

....

The energy, W, used up is equal to $d \times F_D$, where d is the distance traveled. In both cases, $W = 15$ kWh $= 15,000 \times 36000 = 54 \times 10^6$ J.

Let $V_1 = 100$ km/h 27.78 m/s, $V_2 = 60$ km/h 16.67 m/s, $d_1 = 200$ km $= 2 \times 10^5$ m, $V_2 = 362.5$ km $= 3.62.5 \times 10^5$ m.

Then

$$\begin{cases} (a_0 + 27.78^2 a_2) \times 2 \times 10^5 = 54 \times 10^6, \\ (a_0 + 16.67^2 a_2) \times 3.625 \times 10^5 = 54 \times 10^6, \end{cases}$$

or

$$\begin{cases} a_0 + 771.73 a_2 = 270.0 \\ a_0 + 277.89 a_2 = 148.97 \end{cases}$$

The determinant is

$$\Delta = \begin{pmatrix} 1 & 771.73 \\ 1 & 277.89 \end{pmatrix} = 277.89 - 771.3 = -493.41.$$

The numerator of a_2 is

$$N_2 = \begin{pmatrix} 1 & 270.0 \\ 1 & 148.97 \end{pmatrix}.$$

Consequently,

$$A_2 = \frac{-12103}{-493.41} = 0.2453.$$

The aerodynamic drag force is $F = 0.5\rho A C_D V^2 = 0.5 \times 1.29 \times 2V^2 C_D$ $= 1.29 V^2 C_D$. This is, of course, equal to $a_2 V^2$, hence

$$a_2 = 1.29 C_D,$$

$$C_D = \frac{0.2453}{1.29} = 0.19.$$

The aerodynamic drag coefficient of the car is 0.19.

Solution of Problem 15.21

Prob 15.22 The army of Lower Slobovia needs an inexpensive platform for mounting a reconnaissance camcorder that can be hoisted to some height between 200 and 300 m. The proposed solution is a kite that consists of a Göttingen 420 air foil with 10 m^2 of area tethered by means of a 300 m long cable. To diminish the radar signature the cable is a long monocrystal fiber having enormous tensile strength so that it is thin enough to be invisible, offers no resistance to airflow and has negligible weight.

In the theater of operation, the wind speed is a steady 15 m/s at an anemometer height of 12 m, blowing from a 67.5° direction. It is known that this speed grows with height exactly according to the 1/7–power law.

The wing loading (i.e., the total weight of the kite per unit area) is 14.9 kg m^{-2}.

The airfoil has the following characteristics:

$$C_L = 0.5 + 0.056\alpha,$$

$$C_D = 0.05 + 0.012|\alpha|,$$

where α is the attack angle in degrees.

The above values for the lift and drag coefficients are valid in the range $-10^\circ < \alpha < 15^\circ$.

The tethering mechanism is such that the airfoil operates with an angle of attack of 0.

The battlefield is essentially at sea level.

The questions are?

a. Assume that the kite is launched by somehow lifting it to an appropriate height above ground. What is the hovering height?

b. What modifications must be made so that the kite can be launched from ground (i.e, from the height of 12 m). No fair changing the wing loading. Qualitative suggestions with a minimum of calculation are acceptable.

....

Since the attack angle is permanently kept at 0°, the values of C_L and C_D are fixed at

$$C_L = 0.5,$$

$$C_D = 0.05.$$

The lift force is

$$F_L = \tfrac{1}{2}\rho V^2 A C_L, \tag{1}$$

while the drag force is

$$F_D = \tfrac{1}{2}\rho V^2 A C_D. \tag{2}$$

Solution of Problem 15.22

Introducing the known values of A, ρ, C_L and C_D,

$$F_L = 3.225V^2, \tag{3}$$

$$F_D = 0.3225V^2, \tag{4}$$

where V is the effective wind velocity (height dependent),

$$V = V_0 \left(\frac{h}{h_0}\right)^{1/7} = 15 \left(\frac{h}{12}\right)^{1/7} = 10.52h^{1/7}. \tag{5}$$

Gravity exerts a force, $F_G = 14.9Ag = 1462$ N downward on the kite. The net upward force is $F_L - F_G$, while the only horizontal force is F_D. These forces add to a total force, F, that makes an angle γ with the horizontal,

$$\tan\gamma = \frac{F_L - F_G}{F_D} = \frac{3.225V^2 - 1462}{0.3225V^2}$$
$$= \frac{3.225V^2 - 1462}{0.3225V^2} = 10 - 4532V^{-2} = 10 - 40.95h^{-2/7}. \tag{6}$$

Since the tether can only oppose tension (no bending), it must align itself with the direction of the resulting force. γ must be the angle of the tether with the horizontal and the height of the kite is

$$h = 300\sin\gamma. \tag{7}$$

Formally, we have (combining Equations 6 and 7),

$$h = 300\sin[\arctan(10 - 40.95h^{-2/7})],$$

but this is a transcendental equation that must be solved numerically. Take a guess at the value of h. Say $h = 300$ m. Introducing this value into Equation 6, one obtains $\tan\gamma = 1.974$ and $\gamma = 63.133°$.

Introducing this value of γ into Equation 7, one obtains a value $h = 276.6$ which differs from the original guess. Iterate by putting this value again in to Equation 6 and obtaining $\gamma = 59.646°$. This, in turn, yields, $h = 258.88$ m. The next iteration leads to $h = 255.6$ m.

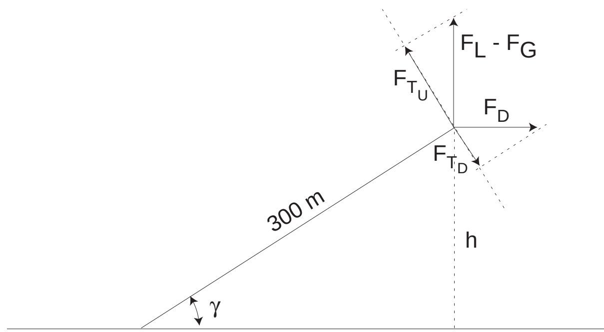
One more iteration will do it! Put the last value of h into Equation 6 and obtain $\gamma = 57.96°$.. This leads to $h = 254.3$ m. Close enough! The correct value is very close to 253 m.

The kite will hover at a 253-m altitude.

If the kite is launched from the ground (assume a height of 12 m), then the wind velocity is 15 m/s. The lift force (Equation 3) will be 725.6 N. This is less than the weight , 1462 N, of the kite and the device will not rise. Notice that if the lift coefficient is increased to $0.5 \times 1462/725.6 = 1.01$, then lift and weight will just balance one another and the slightest upward push will cause a take-off. The desired lift coefficient can be reached if α is increased to 9.06°. This is within the range of validity of the expression for C_L.

....

There is a somewhat different way to look at this problem:



Refer to Figure 1, above. Assume that the kite is at a height, h, above ground. The tether will be at an elevation angle, $\gamma = \arcsin h/300$.

The forces acting on the kite will have a vertical component, $F_L - F_G = 356.9h^{2/7} - 1462$, and a horizontal component, $F_D = 35.69h^{2/7}$.

The vertical component, has itself a component,

$$F_{T_U} = (F_L - F_G)\cos\gamma = (356.9h^{2/7} - 1462)\cos\gamma,$$

perpendicular to the tether. This component torques the tether upwards.

The horizontal component, has a component,

$$F_{T_D} = F_D \sin\gamma = 35.69h^{2/7} \sin\gamma,$$

perpendicular to the tether. This component torques the tether downward.

In steady state, $F_T \equiv F_{T_U} = F_{T_D}$:

$$F_T = (356.9h^{2/7} - 1462)\cos\gamma - 35.69h^{2/7}\sin\gamma. \quad (8)$$

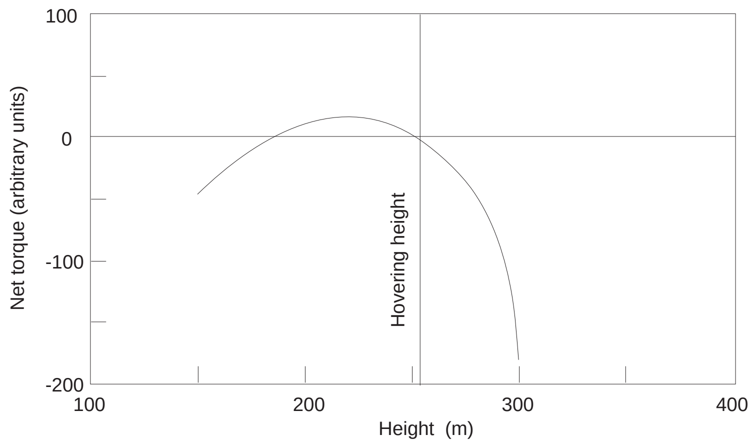
Select an arbitrary value of h, and calculate γ from Equation 7.

Using h and the corresponding γ, calculate F_T from Equation 8.

The result is shown in Figure 2. It can be seen that F_T is initially negative because the lift at low altitudes is smaller than the weight owing to the small wind velocities. Up to a certain height, as h grows, so will F_T. At even higher elevations, F_T starts falling and equilibrium ($F_T = 0$) is reached at about 253 m.

Solution of Problem 15.22

This solution method has the advantage of showing that the hovering height is stable. Indeed, if there is a small perturbation in h (say, h is a little above the correct height), then $F_T < 0$ and the kite comes down.



Solution of Problem 15.22

Prob 15.23 A car massing 1000 kg, has an effective frontal area of 2 m^2. It is driven on a windless day on a flat, horizontal highway (sea level) at the steady speed of 110 km/h. When shifted to neutral, the car will, of course, decelerate and in 6.7 seconds its speed is down to 100 km/h.

From these data, estimate (very roughly) the coefficient of aerodynamic drag of the car. What assumptions and/or simplifications did you have to make to reach such estimate?

Is this estimate of C_D an upper or a lower limit? In other words, do you expect the real C_D to be larger or smaller than the one you estimated?

....

The retarding force on a moving car can , in general, be expressed by

$$F = a_0 + a_1 v + a_2 v^2.$$

The quadratic term is the result of the aerodynamic drag, F_{aer},

$$F_{aer} = \frac{1}{2}\rho v^2 C_D A,$$

where A is the frontal area of the car.

Since insufficient information on the car's performance is available, we will assume that at the, relatively, high speed involved, the total drag is essential equal to the aerodynamic drag.

The deceleration, γ, is then

$$\gamma = mF_{aer}.$$

The deceleration is approximately

$$\gamma = \frac{\Delta v}{\Delta t}.$$

The velocities involved are

110 km/h = 30.56 m/s, and 100 km/h = 27.78 m/s.

The deceleration time is 6.7 seconds.

$$\gamma = \frac{30.56 - 27.78}{6.7} = 0.415 \quad \text{m/s}^2.$$

The mass of the car being 1000 kg, this leads to

$$F_{aer} = 1000 \times 0.415 = 415 = \frac{1}{2}\rho v^2 C_D A$$

$$= \frac{1}{2} \times 1.29 \times 29.17^2 \times 2C_D = 1098C_D.$$

$$C_D = \frac{415}{1098} = 0.38.$$

Here, we used $\rho = 1.29$ (STP) and the average velocity during the deceleration.

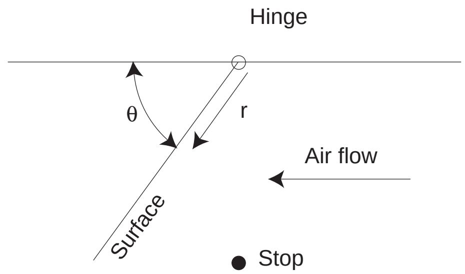
The coefficient of drag of the car is no larger than 0.38.

Since we ignored all other drags, the estimated C_D must be an upper value.

Solution of Problem 15.23

Prob 15.24 In early airplanes, airspeed indicators consisted of a surface exposed to the wind. The surface was attached to a hinge (see figure) and a spring (not shown) torqued the surface so that, in absence of air flow, it would hit a stop and, thus, assume a position with $\theta = 90°$.

The wind caused the surface to move changing θ. This angle, seen by the pilot, was an indication of the air speed.



In the present problem, the surface has dimensions L (parallel to the axis of the hinge) by D (perpendicular to L). $L = 10$ cm, $D = 10$ cm.

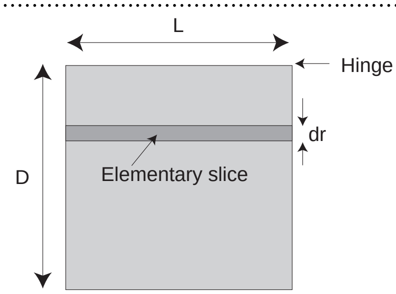
The spring exerts a torque

$$\Upsilon_{spring} = \frac{0.1}{\sin\theta} \quad \text{N m.}$$

The coefficient of drag of the surface is 1.28. Air density is 1.29 kg/m^3.

Calculate the angle, θ, for wind velocities, V, of 0, 10, 20, and 50 m/s.

....



The projected area of the surface is a function of θ because the wind tilts it. The stronger the wind, the smaller the projected area.

Consider a thin slice of the surface as shown in the figure. Its area is

$$dA = Ldr.$$

Solution of Problem 15.24

The force exerted by the wind on this elementary area is

$$dF = \tfrac{1}{2}\rho C_D L \sin\theta V^2 dr = \tfrac{1}{2} \times 1.29 \times 1.28 \times 0.1 \sin\theta V^2 dr$$
$$= 82.6 \times 10^{-3} \sin\theta V^2 dr.$$

The elementary torque from each elementary area is

$$d\Upsilon = rdF,$$

where r is the lever arm, i.e., the distance from the axis of the hinge to the elementary area.

The total torque, Υ_{surf} is

$$\Upsilon_{surf} = \int_0^D rdF = \int_0^D 82.6 \times 10^{-3} \sin\theta V^2 rd$$
$$r = \tfrac{1}{2} 82.6 \times 10^{-3} \sin\theta V^2 D^2 = 413 \times 10^{-6} \sin\theta V^2.$$

Equilibrium is established when $\Upsilon_{surf} = \Upsilon_{spring}$, i.e., when

$$413 \times 10^{-6} \sin\theta V^2 = \frac{0.1}{\sin\theta},$$

from which,

$$\sin\theta = \frac{1}{V}\sqrt{\frac{0.1}{0.000413}} = \frac{15.6}{V}. \qquad (1)$$

We can now tabulate the value of θ versus different values of V:

V	$\theta°$
0	90°
10	90°
20	51.1°
50	18.1°

Remarks:

When $V = 0$, there is no air speed torque and $\theta = 90°$ because the surface hits the stop.

When $V = 10$, Equation 1 leads to $\sin\theta = 1.56$. This is impossible: the air speed torque is still lower than that of the spring and the surface is still held against the stop.

For $V > 15.6$ m/s, Equation 1 leads to the correct value of θ.

Solution of Problem 15.24

Prob 15.25 An EV (electric vehicle) is tested on a horizontal road. The power, P, delivered by the motors is measured in each run which consists of a 2 km stretch covered at constant ground velocity, V. Wind velocity, W, may be different in each run.

Here are the test results:

Run	Wind Direction	Wind Speed (m/s) W	Car Speed (km/h) V	Power (kW) P
1	—	0	90	17.3
2	Head wind	10	90	26.6
3	Head wind	20	90	39.1
4	—	0	36	2.1
5	Tail wind	35	90	4.3

How much power is required to drive this car, at 72 km/h, into a 30 m/s head wind?

....

Let F be the force exerted by the motor on the car. The power it delivers is $P = FV$.

This driving force must be equal to the sum of all retarding forces because the car is moving at constant speed.

The retarding force is a well behaved function of the speed: it may contain a speed independent term, a_0, and a term, $a_1 V$, proportional to the car speed. In addition it will, certainly, contain an aerodynamic drag term, $a_2 U^2$, where U is the induced wind velocity.

When the car is headed into the wind, $U = V + W$.

The car speeds in the table are

10 m/s (36 km/h),

20 m/s (72 km/h, and

25 m/s (90 km/h)

The driving force is

$$F = \frac{P}{V}.$$

$$\frac{P}{V} = a_0 + Va_1 + (V+W)^2 a_2$$

Take Runs 1 and 3:

$$\begin{cases} \frac{17{,}300}{25} = a_0 + 25a_1 + 25^2 a_2 \\ \frac{39{,}100}{25} = a_0 + 25a_1 + 45^2 a_2 \end{cases}$$

$$\begin{cases} 692 = a_0 + 25a_1 + 625a_2 \\ 1564 = a_0 + 25a_1 + 2025a_2 \end{cases}$$

Solution of Problem 15.25

From the above, we obtain

$$a_2 = 0.623,$$

and

$$a_0 + 25a_1 = 303.$$

To separate a_0 from a_1, we have to use a run at a different car speed. Take Run 4:

$$210 = a_0 + 10a_1 + 0.623 \times 10^2,$$

$$a_0 + 10a_1 = 147.7.$$

$$\begin{cases} 303 = a_0 + 25a_1 \\ \\ 148 = a_0 + 10a_1 \end{cases}$$

From which, $a_1 = 10.3$ and $a_0 = 45.5$.

We now have the complete formula for the power necessary to drive the car:

$$P = \left[45.5 + 10.3V + 0.623(V + W)^2\right] V. \tag{1}$$

To make sure that the formula fits the data in the table, we calculated the required power using the formula. This results in

Run	Wind Direction	Wind Speed (m/s) W	Car Speed (km/h) V	Power (kW) P
1	—	0	90	17.3
2	Head wind	10	90	26.6
3	Head wind	20	90	39.1
4	—	0	36	2.1
5	Tail wind	35	90	9.1 ←

Comparing the table above with the one in the problem statement, we see that the powers calculate with Equation 1 match well those in the data. The only exception is in Run 5 (arrow). In this run the tail wind exceeds the car speed and thus the induced wind pushes the car from behind. Clearly the C_D for this situation is different from that when the car moves into the wind.

Having established confidence in Equation 1, we can calculate the power needed to drive the car at 72 km/h into a 30 m/s head wind. We find that 36.2 kW are needed.

For V=20 m/s and W=30 m/s, $P = 36.2$ kW.

Solution of Problem 15.25

Prob 15.26 Here are some data you may need:

Quantity	Earth	Mars	Units
Radius	6.366×10^6	3.374×10^6	m
Density	5,517	4,577	kg/m^3
Surface air pressure	1.00	0.008	atmos.
Surface air temperature	298	190	K
Air composition	20% O_2, 80% N_2	100% CO_2	
Gravitational constant	6.672×10^{-11}		$N\ m^2kg^{-2}$

A parachute designed to deliver a 105 kg load to Mars is tested on Earth when the air temperature is 298 K and the air pressure is 1.00 atmospheres. It is found that it hits the surface with a speed of 10 m/s.

Assume that mass of the parachute itself is negligible. Assume the drag coefficient of the parachute is independent of the density, pressure and temperature of the air.

If we want to have a similar parachute deliver the same load to Mars what must be its area be compared with the area of the test parachute used on Earth?

....

We need to know the acceleration of gravity at the surface of Mars. The gravitational force that a planet of mass, m_{planet}, exerts on an object of mass, m_{object}, on its surface is

$$F_G = G\frac{m_{planet}}{r^2_{planet}}m = G\frac{\rho_{planet}V_{planet}}{r^2_{planet}}m_{object}$$

where ρ is the density and $V = \frac{4}{3}\pi r^3$, is the volume of the planet. Therefore,

$$F_G = G\frac{\rho_{planet}\frac{4}{3}\pi r^3_{planet}}{r^2_{planet}}m_{object} = \frac{4}{3}\pi G\rho_{planet}r_{planet}m_{object}.$$

For Earth,

$$F_G = \frac{4}{3}\pi \times 6.672\times10^{-11}\times 5517\times 6.366\times10^6 m_{object} = 9.816 m_{object} \quad \mathrm{m/s^2}.$$

The coefficient 9.816 in the above equation is, very nearly the correct acceleration of gravity at the surface of the Earth. This confirms our equation.

For Mars,

$$F_G = \frac{4}{3}\pi \times 6.672\times10^{-11}\times 4577\times 3.374\times10^6 m_{object} = 4.316 m_{object} \quad \mathrm{m/s^2}.$$

Thus, the acceleration of gravity at the surface of Mars must be about 4.31 m/s^2.

Next, we must determine the air density at the surface of Mars.

The air density is

$$\rho = nm,$$

Solution of Problem 15.26

where n is the concentration (molecules per cubic meter) and m is the mean molecular mass. But the pressure is given by

$$p = nkT \ \therefore\ n = \frac{p}{kT} \quad \text{and} \quad \rho = \frac{pm}{kT}.$$

The mean molecular mass of Earth's air (20% oxygen, 80% nitrogen) is 28.8 daltons or 47.8×10^{-27} kg. The air density on Earth during the parachute test is

$$\rho = \frac{1.013\times10^5\times47.8\times10^{-27}}{1.38\times10^{-23}\times298} = 1.18 \quad \text{kg/m}^3.$$

Notice that this differs slightly from the air density at STP.

The Martian atmosphere is supposed to be pure CO_2 which has a molecular mass of 44 daltons or $44\times1.66\times10^{-27} = 73.0\times10^{-27}$ kg. The atmospheric pressure is 0.008 atmospheres or 810 Pa. Consequently, the air density is

$$\rho = \frac{810\times73.0\times10^{-27}}{1.38\times10^{-23}\times190} = 0.0225 \quad \text{kg/m}^3.$$

The force that gravity exerts on the parachute is

For Earth:

$$F_{G_{Earth}} = 9.82\times105 = 1031 \quad \text{N}.$$

and for Mars:

$$F_{G_{Mars}} = 4.32\times105 = 454 \quad \text{N}.$$

The air drag on the parachute is

$$F_D = \tfrac{1}{2}\rho V^2 A\ C_D.$$

For Earth:

$$F_D = \tfrac{1}{2}1.18\times10^2 A_{Earth} C_D = 59 A_{Earth} C_D$$

and for Mars:

$$F_D = \tfrac{1}{2}0.0225\times10^2 A_{Mars} C_D = 1.13 A_{Mars} C_D$$

where A_{Earth} and A_{Mars} are the parachute areas for, Earth and for Mars, respectively.

On both planets, $F_G = F_D$,

$$1031 = 59 A_{Earth} C_D \ \therefore\ A_{Earth} C_D = 17.5$$

and

$$454 = 1.13 A_{Mars} C_D \ \therefore\ A_{Mars} C_D = 401.2,$$

$$\frac{A_{Mars}}{A_{Earth}} = \frac{401.2}{17.5} = 22.9$$

The area of the parachute to be used on Mars must be 23 times larger than that of the parachute tested on Earth.

Solution of Problem 15.26

Prob 15.27 An EV experiences an aerodynamic drag of 320 N when operated at sea level (1 atmosphere) and 30 C.

What is the drag when operated at the same speed at La Paz, Bolivia (4000 m altitude, air pressure 0.6 atmospheres) and at a temperature of -15 C?

....

The drags are proportional to the air densities. Hence the drag at La Paz is

$$F_{D_{La\ Paz}} = F_{D_{Sea\ level}} \frac{\rho_{La\ Paz}}{\rho_{Sea\ level}}$$

From the perfect gas law,

$$\rho \propto \frac{p}{RT},$$

hence,

$$\frac{\rho_{La\ Paz}}{\rho_{Sea\ level}} = \frac{p_{La\ Paz} T_{Sea\ level}}{p_{Sea\ level} T_{La\ Paz}} = \frac{0.6}{1} \times \frac{273.3+30}{273.3-15} = 0.70,$$

and

$$F_{D_{La\ Paz}} = 320 \times 0.7 = 224 \quad \text{N}.$$

The air drag at La Paz is 224 N.

Solution of Problem 15.27

Prob 15.28 A trimaran is equipped with a mast on which a flat rigid surface has been installed to act as a sail. This surface is kept normal to the induced wind direction. The boat is 25 km from the shore which is due north of it. A 36 km/h wind, V, blows from south to north. How long will it take to reach the shore if it sails straight down-wind? Ignore any force the wind exerts on the boat except that on the sail.

The area, A, of the sail is 10 m^2.

The coefficient of drag of a flat surface is C_D =1.28.

The air density is $\rho = 1.2$ kg/m^3.

The water exerts a drag force on the trimaran given by

$$F_{water} = 0.5 \times W^2,$$

where, W, is the velocity of the boat relative to the water (there are no ocean currents).

....

The wind velocity is $36,000/3,600 = 10$ m/s.

The induced wind velocity is $V - W$. Consequently the force the wind exerts on the sail is

$$F_{wind} = \frac{1}{2}\rho A C_D (V-W)^2 = \frac{1}{2} \times 1.2 \times 10 \times 1.28 \times (10-W)^2 = 7.68(10-W)^2.$$

Under steady state conditions,

$$F_{wind} = F_{water},$$

$$7.68(10-W)^2 = 0.5W^2,$$

$$100 + W^2 - 20W = \frac{0.5}{7.68}W^2,$$

$$0.9349W^2 - 20W + 100 = 0,$$

$$W = \frac{20 \pm \sqrt{20^2 - 4 \times 100 \times 0.9349}}{2 \times 0.9349} = \begin{cases} 13.42 \\ 7.97 \end{cases} \text{m/s}.$$

The first solution leads to a boat velocity larger than that of the wind which is an impossible situation when a boat runs with a tail wind.

The boat will take 25,000/7.97=3137 seconds (52 minutes and 17 seconds) to reach the shore.

Solution of Problem 15.28

Prob 15.29 Two identical wind turbines are operated at two locations with the following wind characteristics:

<u>Location 1</u>

Percent of time	Wind speed
50	**10 m/s**
30	**20 m/s**
20	**25 m/s**

<u>Location 2</u>

Percent of time	Wind speed
50	**15 m/s**
50	**21 m/s**

Which wind turbine generates more energy? What is the ratio of energy generated by the two wind turbines?

....

<u>Location 1</u>

The energy generated is proportional to

$$W_1 \propto 0.5 \times 10^3 + 0.3 \times 20^3 + 0.2 \times 25^3 = 6025.$$

<u>Location 2</u>

The energy generated is proportional to

$$W_1 \propto 0.5 \times 15^3 + 0.5 \times 21^3 = 6318.$$

Ratio: $\frac{\text{Location2}}{\text{Location1}} = \frac{6318}{6025} = 1.05.$

Location 2 produces 5% more energy than Location 1.

Solution of Problem 15.29

Prob 15.30 What is the air density of the planet in Problem 1.22 if the temperature is 450 C and the atmospheric pressure is 0.2 MPa?

....

We need to know the specific volume of one kilomole of gas:

$$V = \frac{RT}{p} = \frac{8314 \times (450 + 273)}{2 \times 10^5} = 30.0 \quad \text{m}^3.$$

The density is

$$\rho = \frac{0.3 \times 17 + 0.5 \times 44 + 0.2 \times 28}{30.0} = 1.09 \quad \text{kg/m}^3.$$

The air density is 1.09 kg/m^3.

Solution of Problem 15.30

Prob 15.31 One may wonder how an apparently weak effect (the reduction of pressure on top of an airfoil caused by the slightly faster flow of air) can lift an airplane.

Consider a Cessna 172 (a small 4-seater). It masses 1200 kg and has a total wing area of 14.5 m^2. In horizontal flight at sea level, what is the ratio of the average air pressure under the wing to the pressure above the wing?

....

The **wing loading**, Γ, of the Cessna 172 is

$$\Gamma = \frac{1200}{14.5} \approx 83 \quad \text{kg/m}^2. \tag{1}$$

Each square meter of wing surface must lift 83 kg. The pressure exerted by the air pushing *up* the underside of the wing is about 1 kg/cm^2 (1 atmosphere). This amounts to 10^4 kg/m^2.

To lift the plane, the pressure pushing *down* on the top of the wing must be $10,000 - 83 = 9,917$ kg/m^2. Thus the pressure ratio is $10,000/9,917 = 1.0084$.

The pressures under and on top of the wing are almost the same. Their ratio is 1.0084:1.

Solution of Problem 15.31

Prob 15.32 A car has the following characteristics:
Mass, m, = 1,200 kg.
Frontal area, A, = 2.2 m^2.
Coefficient of drag, C_D, = 0.33.
The experiment takes place under STP conditions.

When placed on a ramp with a $\theta = 1.7°$ angle, the car (gears in neutral, no brakes) will, of course, start moving and will accelerate to a speed of 1 m/s. This speed is maintained independently of the length of the ramp. In other words, it will reach a terminal velocity of 1 m/s.

When a steeper ramp is used ($\theta = 2.2°$), the terminal speed is 3 m/s.

Now place the car on a horizontal surface under no wind conditions. Accelerate the car to 111.60 km/h and set the gears to neutral. The car will coast and start decelerating. After a short time, Δt, the car will have reached the speed of 104.4 km/h.

What is the value of Δt?

....

Assume that the force retarding the motion of the car is given by the power series,

$$F = a_0 + a_1 V + a_2 V^2. \tag{1}$$

The quadratic term in Equation 1, represents the aerodynamic drag and, thus, the coefficient, a_2, is

$$a_2 = \frac{1}{2}\rho A C_D = \frac{1}{2} 1.29 \times 2.2 \times 0.33 = 0.468. \tag{2}$$

If the car is on a ramp, its weight, $F_g = 9.81m$, has a component along the ramp (i.e., a component impelling the car forward),

$$F_{g_f} = F_g \sin\theta = 9.81 \times 1200 \sin\theta = 11,772 \sin\theta. \quad \text{N} \tag{3}$$

Thus for the two different ramp slopes,

$$\begin{cases} 11,772 \sin 1.7° = 349.2 = a_0 + 1 \times a_1 + 1^2 \times 0.468, & (4) \\ 11,772 \sin 2.2° = 451.9 = a_0 + 3 \times a_1 + 3^2 \times 0.468. & (5) \end{cases}$$

$$\begin{cases} 348.7 = a_0 + 1 \times a_1, & (6) \\ 447.8 = a_0 + 3 \times a_1. & (7) \end{cases}$$

The determinant of these two simultaneous equations is

$$\Delta = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = 2. \tag{8}$$

Solution of Problem 15.32

The numerators are

$$N_0 = \begin{pmatrix} 348.7 & 1 \\ 447.8 & 3 \end{pmatrix} = 598.3 \tag{9}$$

and

$$N_1 = \begin{pmatrix} 1 & 348.7 \\ 1 & 447.8 \end{pmatrix} = 99.1 \tag{10}$$

Hence

$$a_0 = \frac{598.3}{2} = 299.2, \tag{11}$$

$$a_1 = \frac{99.1}{2} = 49.6, \tag{12}$$

Equation 1 now becomes,

$$F = 299.2 + 49.6V + 0.468V^2. \tag{13}$$

At (111.6+104.4)/2=108 km/h, i.e., at exactly 30 m/s, the mean of the two velocities mentioned in the question, the retarding force on the car is

$$F = 299.2 + 49.6 \times 30 + 0.468 \times 30^2 = 2138.4 \quad \text{N}. \tag{14}$$

The power associated with this retarding force is

$$P = Fv = 2138.4 \times 30 = 64,152 \quad \text{W}. \tag{15}$$

The change in kinetic energy when the car decelerates from 111.6 km/h (31 m/s) to 104.4 km/h (29 m/s) is

$$\Delta U = \tfrac{1}{2}m\Delta V = \tfrac{1}{2} \times 1200 \times (31^2 - 29^2) = 72,000 \quad \text{J}. \tag{16}$$

But $P = \Delta U/\Delta t$, hence

$$\Delta t = \frac{\Delta U}{P} = \frac{72,000}{64,152} = 1.12 \quad \text{m/s}. \tag{17}$$

****

Another approach is given below:
The deceleration is

$$\gamma = \frac{F}{m} = \frac{2138.4}{1200} = 1.78 \quad \text{m/s}^2. \tag{18}$$

The car slows down by 2 m/s (from 31 to 29 m/s). Hence, the time for this is

$$\Delta T = \frac{2 \text{ m/s}}{1.78 \text{ m/s}^2} = 1.12 \quad \text{seconds}. \tag{19}$$

The car takes 1.12 seconds to slow from 111.6 to 104.4 km/h.

Solution of Problem 15.32

Prob 15.33 The observed efficiency of a "gyromill" type wind turbine is given by

$\eta = 0$, **for** $\frac{U}{V} \leq 2$,
$\eta = 0.280\left(\frac{U}{V} - 2\right)$, **for** $2 \leq \frac{U}{V} \leq 5$,
$\eta = -0.420\frac{U}{V} + 2.940$, **for** $\frac{U}{V} > 5$.

The turbine has 2 blades or wings each 30 m long and the radius of the device is 9 m.

When operating at sea level under a uniform 15-m/s wind what power does it deliver to a load whose torque is 50,000 Nm independently of the rotational speed? What is the rotation rate of the turbine (in rpm)?

....

The power that the wind turbine delivers is

$$P_D = \frac{16}{27} \times \tfrac{1}{2}\rho V^3 \times 2rH\eta = \frac{16}{27} \times \tfrac{1}{2} \times 1.29 \times 15^3 \times 2 \times 9 \times 30\eta = 696,600\eta.$$

The angular velocity is

$$\omega = \frac{U}{r} = \frac{U}{V} \times \frac{V}{R} = \frac{U}{V} \times \frac{15}{9} = 1.667\frac{U}{V}.$$

The torque of the wind turbine is

$$\Upsilon = \frac{P_D}{\omega} = \frac{696,600\eta}{1.667U/V} = \frac{418,000\eta}{\frac{U}{V}}.$$

If $\frac{U}{V} \leq 2$, the torque of the wind turbine is zero, because $\eta = 0$.
If $2 \leq \frac{U}{V} \leq 5$, the torque is

$$\Upsilon = \frac{418,000 \times 0.28\left(\frac{U}{V} - 2\right)}{\frac{U}{V}} = \Upsilon_{Load} = 50,000.$$

Thus, $\frac{U}{V} = 3.492$ and $\eta = 0.418$.
If $\frac{U}{V} > 5$,

$$\Upsilon = \frac{418,000 \times (-0.420\frac{U}{V} + 2.940)}{\frac{U}{V}} = 50,000.$$

Thus, $\frac{U}{V} = 5.448$ and $\eta = 0.652$.
Only this second equilibrium point is stable. Consequently,

$$\omega = 1.667 \times 5.448 = 9.08 \quad \text{rad/sec.}$$

Solution of Problem 15.33

The power generated is

$$P_D = \Upsilon * \omega = 50,000 * 9.08 = 454,000 \quad \text{W.}$$

The wind turbine generates 454 kW.

The rotation rate is

$$\text{rpm} = 9.08 \frac{\text{radians}}{\text{second}} \times \frac{1}{2\pi} \frac{\text{rotations}}{\text{radian}} \times 60 \frac{\text{seconds}}{\text{minute}} = 86.7.$$

The wind turbine rotates at 86.7 rpm.

Solution of Problem 15.33

Prob 15.34 A standard basket ball has a radius of 120 mm and a mass of 560 grams. Its coefficient of drag, C_D, is 0.3 (a wild guess), independently of air speed.

Such a ball is dropped from an airplane flying horizontally at 12 km altitude over the ocean. What is the velocity of the ball at the moment of impact on the water?

Make reasonable assumptions.

....

The ball characteristics expressed in the SI are,
$r = 0.12$ m,
$m = 0.56$ kg,
$C_D = 0.3$.
The cross-sectional area is

$$A = \pi \times r^2 = 0.045 \quad \text{m}^2.$$

The drag exerted by the air flow is

$$F_D = \frac{1}{2}\rho V^2 C_D A = \frac{1}{2} \times 1.29 \times 0.3 \times 0.045 V^2 = 0.0087 V^2 \quad \text{N}$$

This assumes an air density of 1.29 kg/m^2.
The force of gravity on the ball is

$$F_g = mg = 0.56 \times 9.81 = 5.49 \quad \text{N}.$$

In falling through the air, the ball will very quickly reach a terminal velocity in which the drag force exactly balances the gravitational attraction.

$$F_D = F_g,$$

$$0.02V^2 = 5.49,$$

$$V = \sqrt{\frac{5.49}{0.0087}} = 25.1 \quad \text{m/s}$$

The basket ball will impact the sea at 25 m/s.

The main assumptions made were:

1. The ball reaches terminal velocity before hitting the ocean.
2. The horizontal velocity of the ball (owing to the motion of the airplane has been reduced to zero by the air drag.
2. The air density at impact is 1.29 kg/m^3.

Solution of Problem 15.34

The assumptions above are the result of common sense. However, although not required in this problem, if one is really skeptical one can come up with a more rigorous solution.

Assume again that, at sea level, $\rho = 1.29\ \text{kg/m}^3$, and that the atmosphere is isothermal at 300 K. If so, the air density at any height is

$$\rho = 1.29 \exp\left(-\frac{h}{8800}\right).$$

The acceleration of gravity is $9.81\ \text{m/s}^2$ at sea level. Consequently at 12, 000 height it is

$$g = 9.81\left(\frac{6366}{6366+12}\right)^2 = 9.77 \quad \text{m/s}^2,$$

where 6366 km is the radius of Earth. The change in the acceleration of gravity is small enough to be neglected.

One must now perform an integration starting at 12 km height and an initial vertical velocity $V_0 = 0$.

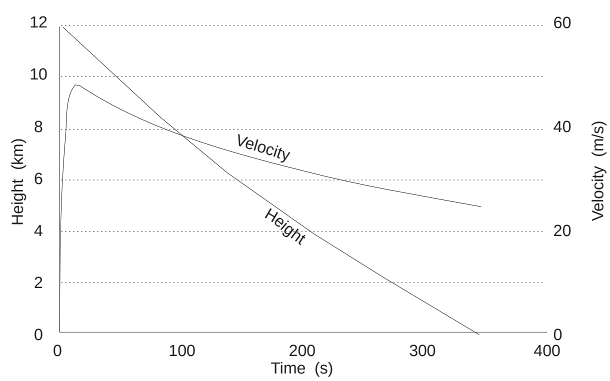
The velocity after Step i is

$$V_i = V_{i-1} + \gamma_i \Delta t,$$

where Δt is the integration time interval. If the mass of the ball is m and the force that acts (vertically) on it is F, then

$$\gamma_i = \frac{F_i}{m} = 9.81 - \frac{F_{D_i}}{m} = 9.81 - \frac{\frac{1}{2}\rho V_{i-1}^2 C_D A}{m} = 9.81 - 0.0121\rho_{i-1}V_{i-1}^2$$
$$= 9.81 - 0.0155 \exp -\left(\frac{h_i}{8800}\right) V_{i-1}^2.$$

$$h_i = h_{i-1} - V_i \Delta t.$$



Solution of Problem 15.34

The figure shows that the ball reaches its terminal velocity of 48.2 m/s in 14 seconds, having fallen 521 m to an altitude of 11,479 m. From this moment on, the velocity diminishes owing to the increasing air density, decreasing to 25.2 m/s on impact.

But, it is not necessary to use all tis rigor. One can make some simple estimates of when the terminal velocity is reached.

At 12 km, $\rho = 1.29 \exp\left(-\frac{12000}{8800}\right) = 0.33$ kg/m^3. Gravity exerts a force of $0.56 \times 9.8 = 5.49$ N. Terminal velocity is achieved when the drag equals the weight:

$$\frac{1}{2}\rho C_d A V^2 = \frac{1}{2} 0.33 \times 0.3 \times 0.045 V^2 = 0.0022 V^2 = 5.49,$$

$$V = \sqrt{\frac{5.49}{0.0022}} = 49.7 \quad \text{m/s}$$

This is close to the exact value of 48.2 m/s.

The average acceleration is $9.8/2 = 4.9$ m/s^2 suggesting that it takes some 10 seconds (actually 14 s) to reach the terminal velocity.

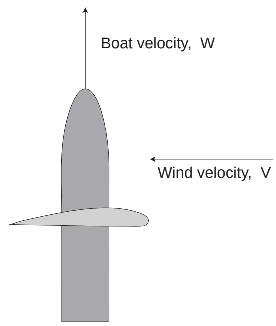
The average velocity during the acceleration is 24.8 m/s and the corresponding vertical distance 248 m. This is a bit off but indicates that the terminal velocity is reached at very great altitude.

Solution of Problem 15.34

Prob 15.35 This is a terrible way to sail a ship, but leads to a simple problem.

In the absence of wind relative to the boat, a boat's engine power, P_{Eng}, of 20,680 W is needed to maintain a speed of 15 knots (1 knot is 1852 meters per hour). The efficiency of the propeller is 80%. Assume that water drag is proportional to the water speed squared.

Under similar conditions, only 45 W are needed to make the boat move at 1 m/s.



This very boat, is now equipped with an 10 m^2 airfoil, mounted vertically and oriented perpendicularly to the boat's axis. See figure.

The coefficient of lift of the airfoil is

$$C_L = (0.05\alpha + 0.5)$$

and is valid for $-10 < \alpha < 10$. In these two equations, α is in degrees.

The airfoil exerts a lift that, it is to be hoped, propels the boat due north when a 15 knot wind blows from the east.

What is the speed of the boat?

....

15 knots corresponds to $15 \times 1,852 = 27,780$ m/hour or $27,780/3,600 = 7,717$m/s. The power generated by the propeller (screw) is $P_p = 0.8 P_{Eng} = 0.8 \times 20,680 = 16,544$ W. The resulting force, F_p, generated by the propeller is

$$F_p = \frac{P_p}{W,}$$

where W is the speed of the boat relative to the water.

$$F_p = \frac{16,544}{W} = aW^2.$$

Here, a is a coefficient of proportionality.

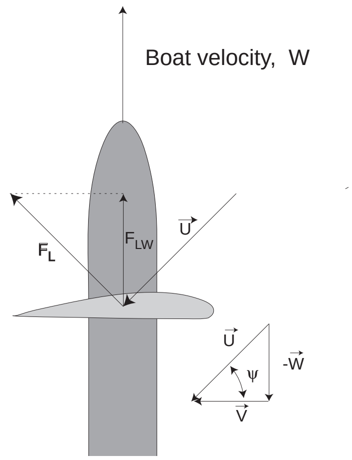
$$a = \frac{16,544}{W^3} = \frac{16,544}{7.717^3} = 36$$

Repeating the calculation for a speed of 1 m/s,

$$a = 0.8 \times \frac{45}{1^3} = 36,$$

just as in the previous case.

Solution of Problem 15.35



The combination of the boat velocity induced wind, $-\vec{W}$, and the real wind, $\vec{V}$, causes an apparent velocity, $\vec{U}$, to act on the airfoil, creating a lift force, F_L.

$$U = \sqrt{V^2 + W^2}.$$

$$\cos\psi = \frac{V}{U} = \frac{V}{\sqrt{V^2 + W^2}}.$$

Notice that the angle of attack, α, is equal to $-\psi$.

Consequently, the lift force is

$$F_L = \tfrac{1}{2}\rho U^2 A\ C_D = \tfrac{1}{2}\rho(V^2 + W^2)A(-0.05\arccos\frac{V}{\sqrt{V^2 + W^2}} + 0.5)$$

The component of this lift force along the direction of the motion of the boat is

$$\begin{aligned} F_{LW} &= F_L \cos\psi \\ &= \tfrac{1}{2}\rho(V^2 + W^2)A(-0.05\arccos\frac{V}{\sqrt{V^2 + W^2}} + 0.5)\frac{V}{\sqrt{V^2 + W^2}} \\ &= \tfrac{1}{2}\rho V\sqrt{(V^2 + W^2)}A(-0.05\arccos\frac{V}{\sqrt{V^2 + W^2}} + 0.5). \end{aligned}$$

This must equal to the drag that the water exerts on the boat,

$$aW^2 = \tfrac{1}{2}\rho V\sqrt{(V^2 + W^2)}A(-0.05\arccos\frac{V}{\sqrt{V^2 + W^2}} + 0.5).$$

Introducing the known values,

$$W^2 = 1.383\sqrt{7.717^2 + W^2}\left(-0.05\arccos\frac{7.717}{\sqrt{7.717^2 + W^2}} + 0.5\right)$$

The only unknown in the above equation is W. A numerical solution yields $W = 1.068$ m/s.

The boat will move at a speed of 1.068 m/s or 2.08 knots.

Solution of Problem 15.35

Prob 15.36 A sail plane (a motorless glider) is at a 500 m altitude and is allowed to glide down undisturbed. The atmosphere is perfectly still (no winds, no thermals [vertical winds]). Air temperature is 0 C and air pressure is 1 atmosphere.

The wings have a 20 m^2 area and at their lift coefficient is $C_L = 0.5$ and their drag coefficient is $C_D = 0.05$. Assume, to simplify the problem, that the rest of the sail plane (fuselage, empennage, etc.) produces no lift and no drag. The whole machine (with equipment and pilot) masses 600 kg.

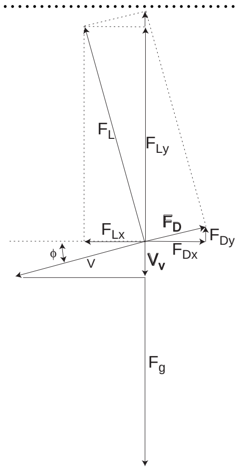
Naturally, as the sail plane moves forward, it looses some altitude. The *glide ratio* is defined as the ratio of distance moved forward to the altitude lost.

a. What is the glide ratio of this sail plane?
b. What is the forward speed of the plane?
c. To keep the plane flying as described, a certain amount of power is required. Where does this power come from and how much is it?

....

a. What is the glide ratio of this sail plane?

....



Refer to the figure.

The sail plane is diving with a velocity, V, making an angle, ϕ, with the horizontal. A lift, F_L, and a drag, F_D, are generated. The horizontal component of the lift is F_{L_x}, and that of the drag is F_{D_x}. Since the plane is not accelerating, the horizontal forces acting on it must be zero:

$$F_{L_x} = F_{D_x},$$

$$\frac{1}{2}\rho V^2 A\ C_L \sin\phi = \frac{1}{2}\rho V^2 A\ C_D \cos\phi,$$

Solution of Problem 15.36

$$\tan\phi = \frac{C_D}{C_L} = \frac{0.05}{0.5} = 0.1,$$

$$\phi = 5.71^\circ.$$

The glide ratio is $V\cos\phi{:}V\sin\phi$ or 10:1.

The glide ratio is 10:1.

---

b. What is the forward speed of the plane?

....

There are two forces that lift the plane: $F_{L_y} = F_L\cos\phi$ and $F_{D_y} = F_D\sin\phi$. These forces must exactly balance the force of gravity, $F_g = mg$, because the plane is not accelerated.

$$F_L\cos\phi + F_D\sin\phi = mg = \tfrac{1}{2}\rho V^2 A(C_L\cos\phi + C_D\sin\phi).$$

$$V = \sqrt{\frac{2mg}{\rho A}} \times \frac{1}{\sqrt{C_L\cos\phi + C_D\sin\phi}}$$
$$= 21.36 \times \frac{1}{0.5\cos 5.71^\circ + 0.05\sin 5.71^\circ} = 30.13 \quad \text{m/s.}$$

The speed of the plane is 30.13 m/s or 108.5 km/hr.

---

c. To keep the plane flying as described, a certain amount of power is required. Where does this power come from and how much is it?

....

The power comes from the rate of change of the plane's potential energy (because it is loosing altitude).

$$P = mgV_v.$$

The vertical velocity, V_v, is

$$V_v = V\sin\phi = 30.13\sin 5.71^\circ = 3.01 \quad \text{m/s.}$$

$$P = 600 \times 9.81 \times 3.0 = 17,600 \quad \text{W.}$$

The plane uses 17.6 kW.

Solution of Problem 15.36

Prob 15.37 The drag force on a car can be expressed as power series in v, the velocity of the car (assuming no external wind):

$$F_D = a_0 + a_1 v + a_2 v^2. \tag{1}$$

For simplicity, assume $a_0 = 0$.

A car drives 50 km on a horizontal road (at sea level) at a steady speed of 60 km/h. Careful measurements show that a total of 1.19×10^7 J were used. Next, the car drives another 50 km at a speed of 120 km/h and uses 3.10×10^7 J. The frontal area of the car is 2.0 m^2. What is the coefficient of drag. C_D, of the car?

....

The first part of the trip lasted

$$t_1 = \frac{50}{60} = 0.833 \quad \text{hr} \quad \text{or} \quad 3000 \quad \text{s}. \tag{2}$$

The speed was

$$v_1 = 60 \quad \text{km/hr} = 16.67 \quad \text{m/s}. \tag{3}$$

The power used was

$$P_1 = F_{D_1} v = a_1 v^2 + a_2 v^3. \tag{4}$$

The energy used up is

$$\begin{aligned} W_1 = & P_1 t = (a_1 v^2 + a_2 v^3) = (16.67^2 a_1 + 16.67^3 a_2)3000 \\ & = 833.3 \times 10^3 a_1 + 13.89 \times 10^6 a_2 = 1.19 \times 10^7. \end{aligned} \tag{5}$$

Repeating all of the above for the second part of the trip,

$$W_2 = 1.666 \times 10^6 a_1 + 55.54 \times 10^6 a_2 = 3.10 \times 10^7. \tag{6}$$

Simultaneous solution of Equations 5 and 6 yields

$$\begin{cases} a_1 = 9.95 \\ a_2 = 0.260. \end{cases}$$

From aerodynamic considerations, the coefficient, a_2, must be

$$a_2 = \tfrac{1}{2} \rho A C_D, \tag{7}$$

hence

$$C_D = \frac{2a_2}{\rho A} = \frac{2 \times 0.260}{1.29 \times 2} = 0.20. \tag{8}$$

The coefficient of drag is 0.2.

Solution of Problem 15.37

Prob 15.38

Percentage of time	m/s
10	Calm
20	5
40	10
30	15

The wind statistics (over a whole year) at a given site are as shown in the table.

When the wind has a speed of 15 m/s, the wind turbine delivers 750 kW. What is the number of kWh generated in a one year period?

....

The cube of the cubic mean wind velocity is

$$< v >^3 = 0.2 \times 5^3 + 0.4 \times 10^3 + 0.3 \times 15^2 = 1.438 \times 10^3 \quad \mathrm{m^3/s^3}. \tag{1}$$

The average power generated must be

$$P_{ave} = 750\frac{1438}{15^3} = 319.6 \quad \mathrm{kW}. \tag{2}$$

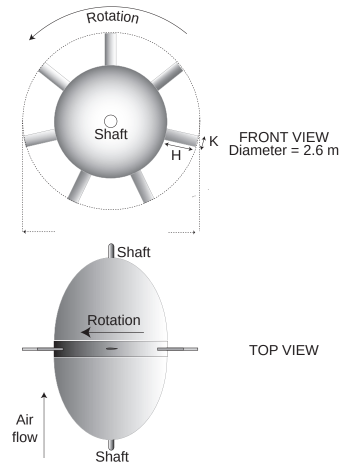
The number of hours in a year is $365 \times 24 = 8760$, thus, the energy generated in one year is

$$W = 8760 \times 319.6 = 2.8 \times 10^6 \quad \mathrm{kWh/year}. \tag{3}$$

The wind turbine generates 2.8 GWh/year.

Solution of Problem 15.38

Prob 15.39



Consider a turbine consisting of seven blades each shaped as a NACA W1 symmetric airfoil with a 32 cm chord, K, and sticking out 52 cm, H, above a fairing. These blades have their reference plane aligned with the plane of rotation of the turbine. The diameter is 2.6 m. The device spins at 1050 rpm in such a direction that, if the turbine rotates in calm air, the angle of attack is zero.

To an acceptable approximation, the coefficients of the airfoil are:

$$C_L = 0.08\alpha - 0.0001\alpha^3$$

For $|\alpha| < 11^\circ$:

$$C_D = 0.0062 \exp(0.2|\alpha|)$$

For $11^\circ < |\alpha| < 21^\circ$:

$$C_D = -0.415 + 0.0564|\alpha| - 0.001 * |\alpha|^2.$$

In the formulas above, α is in degrees.

The air stream that drives the turbine flows vertically up in the drawing (it flows parallel to the turbine shaft) and has a density 3 times that of the air at STP. It's velocity is 28.6 m/s.

How much power does the turbine deliver to the shaft?

Repeat for an identical airflow moving in the opposite direction.

....

Solution of Problem 15.39

The angular velocity of the foil is

$$\omega = 2\pi\frac{rpm}{60} = 2\pi\frac{1050}{60} = 110 \quad \text{rad/s.} \tag{1}$$

The linear velocity of the air foil is:

$$U = \omega r, \tag{2}$$

where r is the radial distance of the point considered from the axis of the turbine and $rps = rpm/60 = 1050/60 = 17.5$ rotations per second. Thus,

$$U = 2\pi \times 17.5r = 110r \quad \text{m/s.} \tag{3}$$

The induced wind velocity, W (the wind perceived by the airfoil), is

$$\vec{W} = \vec{U} + \vec{V} \tag{4}$$

$$W = \sqrt{U^2 + V^2} = \sqrt{12,100r^2 + 28.6^2} \quad \text{m/s.} \tag{5}$$

The angle between $\vec{W}$ and $\vec{U}$ is, in this case the angle of attack, α.

$$\alpha = \arctan\frac{V}{U} = \arctan\frac{28.6}{110r} = \arctan\frac{0.26}{r}.$$

This means that $\sin\alpha = \frac{1}{\sqrt{1+r^2/0.26}}$ and $\cos\alpha = \frac{1}{\sqrt{1+0.26^2/0.26r^2}}$

Notice that $0.78 < r < 1.30$. Thus, $18.41 < \alpha < 11.31$. Consequently, α is always larger than $11°$ and the appropriate formula for C_D is

$$C_D = -0.415 + 0.0564|\alpha| - 0.001 * |\alpha|^2. \tag{7}$$

The aerodynamic coefficients for the airfoil are

$$C_L = 0.08 \times \arctan\frac{0.26}{r} - 0.0001 \times \left(\arctan\frac{0.26}{r}\right)^3 \tag{8}$$

$$C_D = -0.415 + 0.0564 \times \left|\arctan\frac{0.26}{r}\right| - 0.001 \times \left(\arctan\frac{0.26}{r}\right)^2. \tag{9}$$

The density of air at STP is 1.29 kg/m^3. In this problem, it is $3 \times 1.29 = 3.87$ kg/m^3. The dynamic pressure, p_{dyn}, is

$$p_{dyn} = \tfrac{1}{2}\rho W^2 = \tfrac{1}{2} \times 3.87 \times (12,100r^2 + 28.6^2) = 23,410r^2 + 1583 \quad \text{N.} \tag{10}$$

The elementary area of each blade is

$$dA = Kdr = 0.32dr \quad \text{m}^2. \tag{11}$$

The lift force generated by each elementary area of blade is

$$dF_L = p_{dyn}C_L dA = 0.32p_{dyn}C_L dr. \tag{12}$$

Solution of Problem 15.39

The drag force generated by each elementary area of blade is

$$dF_D = p_{dyn}C_D dA = 0.32 p_{dyn} C_D dr. \tag{13}$$

The elementary torquing force is

$$dF_T = dF_L \sin\alpha - dF_D \cos\alpha = 0.32 p_{dyn}(C_L \sin\alpha - C_D \cos\alpha)dr. \tag{14}$$

The elementary torque is

$$d\Upsilon = r dF_T = 0.32 p_{dyn}(C_L \sin\alpha - C_D \cos\alpha) r dr. \quad (15$$

and the total torque per blade is

$$\Upsilon = \int_{0.78}^{1.30} d\Upsilon \tag{16}$$

The power each blade delivers is

$$P_b = \Upsilon\omega = 110\Upsilon \quad \text{W.} \tag{17}$$

The power delivered by all seven blades of the turbine is

$$P = 7 \times 110\Upsilon = 770\Upsilon \quad \text{W.} \tag{18}$$

To obtain a numerical answer, it is necessary to evaluate the integral in Equation (16). The easiest way to accomplish this is to do a numerical integration using a spread sheet. The table on the next page illustrates the integration using an steps $dr = 0.01$ m. Coarser intervals will yield different results.

The integration yields $\Upsilon = 183.435$ N m for the torque owing to a single blade. The torque of all 7 blades amounts to1284 N m. The corresponding power, $\omega\Upsilon$, is 141.2 kW.

The turbine delivers 141 kW. Thanks to the symmetry of the turbine, when the airflow is reversed, the turbine still rotates in the original direction and delivers the same power.

Solution of Problem 15.39

r (m)	Alpha (rad)	Alpha (deg)	C_L	C_D	p_{dyn} (Pa)	dF_T (N)	$d\Upsilon$ (N m)	Υ (N m)
1.30	0.1974	11.31	0.7601	0.0950	41,146	7.367	9.577	9.577
1.29	0.1989	11.40	0.7637	0.0978	40,540	7.131	9.199	18.776
1.28	0.2004	11.48	0.7672	0.1007	39,938	6.899	8.831	27.606
1.27	0.2019	11.57	0.7707	0.1037	39,341	6.673	8.474	36.081
1.26	0.2035	11.66	0.7742	0.1066	38,749	6.451	8.128	44.209
1.25	0.2051	11.75	0.7778	0.1096	38,161	6.234	7.792	52.001
1.24	0.2067	11.84	0.7813	0.1127	37,578	6.021	7.466	59.467
1.23	0.2083	11.94	0.7848	0.1157	37,000	5.814	7.151	66.618
1.22	0.2100	12.03	0.7883	0.1188	36,426	5.610	6.845	73.463
1.21	0.2117	12.13	0.7918	0.1219	35,858	5.412	6.548	80.011
1.20	0.2134	12.23	0.7953	0.1250	35,293	5.218	6.261	86.273
1.19	0.2151	12.32	0.7988	0.1282	34,734	5.028	5.983	92.256
1.18	0.2169	12.43	0.8022	0.1314	34,179	4.843	5.714	97.971
1.17	0.2187	12.53	0.8056	0.1347	33,629	4.662	5.454	103.425
1.16	0.2205	12.63	0.8090	0.1379	33,083	4.485	5.203	108.628
1.15	0.2223	12.74	0.8124	0.1412	32,543	4.313	4.959	113.587
1.14	0.2242	12.85	0.8157	0.1445	32,007	4.144	4.724	118.311
1.13	0.2262	12.96	0.8191	0.1479	31,475	3.980	4.497	122.808
1.12	0.2281	13.07	0.8223	0.1513	30,949	3.819	4.278	127.086
1.11	0.2301	13.18	0.8255	0.1547	30,426	3.663	4.066	131.152
1.10	0.2321	13.30	0.8287	0.1582	29,909	3.510	3.861	135.013
1.09	0.2342	13.42	0.8318	0.1617	29,396	3.361	3.664	138.677
1.08	0.2362	13.54	0.8349	0.1652	28,888	3.216	3.473	142.150
1.07	0.2384	13.66	0.8379	0.1688	28,385	3.074	3.290	145.440
1.06	0.2405	13.78	0.8408	0.1723	27,886	2.936	3.112	148.552
1.05	0.2427	13.91	0.8436	0.1760	27,393	2.801	2.941	151.493
1.04	0.2450	14.04	0.8464	0.1796	26,903	2.670	2.776	154.270
1.03	0.2473	14.17	0.8490	0.1833	26,419	2.541	2.617	156.887
1.02	0.2496	14.30	0.8516	0.1870	25,939	2.416	2.464	159.351
1.01	0.2520	14.44	0.8540	0.1908	25,464	2.293	2.316	161.667
1.00	0.2544	14.57	0.8564	0.1946	24,993	2.173	2.173	163.840
0.99	0.2568	14.72	0.8586	0.1984	24,527	2.056	2.036	165.876
0.98	0.2593	14.86	0.8606	0.2022	24,066	1.942	1.903	167.779
0.97	0.2619	15.00	0.8626	0.2061	23,609	1.830	1.775	169.554
0.96	0.2645	15.15	0.8643	0.2100	23,158	1.720	1.651	171.205
0.95	0.2671	15.31	0.8659	0.2140	22,711	1.612	1.531	172.736
0.94	0.2698	15.46	0.8673	0.2180	22,268	1.506	1.416	174.151
0.93	0.2726	15.62	0.8685	0.2220	21,830	1.402	1.304	175.455
0.92	0.2754	15.78	0.8695	0.2260	21,397	1.299	1.195	176.650
0.91	0.2783	15.95	0.8702	0.2301	20,969	1.198	1.090	177.740
0.90	0.2812	16.11	0.8707	0.2342	20,545	1.098	0.988	178.728
0.89	0.2842	16.28	0.8709	0.2383	20,126	0.999	0.889	179.617
0.88	0.2873	16.46	0.8708	0.2424	19,712	0.900	0.792	180.410
0.87	0.2904	16.64	0.8705	0.2466	19,302	0.802	0.698	181.108
0.86	0.2936	16.82	0.8697	0.2508	18,897	0.705	0.606	181.714
0.85	0.2968	17.01	0.8686	0.2550	18,497	0.607	0.516	182.230
0.84	0.3002	17.20	0.8672	0.2592	18,101	0.509	0.428	182.657
0.83	0.3036	17.39	0.8653	0.2635	17,710	0.411	0.341	182.998
0.82	0.3070	17.59	0.8629	0.2677	17,324	0.311	0.255	183.253
0.81	0.3106	17.80	0.8601	0.2720	16,942	0.211	0.171	183.424
0.80	0.3142	18.00	0.8567	0.2763	16,565	0.108	0.087	183.511
0.79	0.3179	18.22	0.8528	0.2806	16,193	0.004	0.004	183.514
0.78	0.3218	18.43	0.8483	0.2849	15,826	-0.102	-0.079	183.435

When doing the numerical integration it will become apparent that the

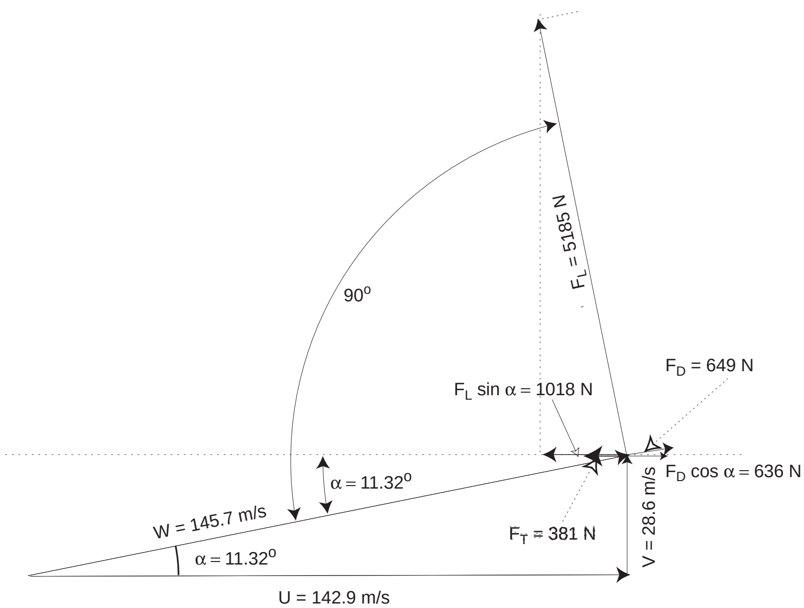
Solution of Problem 15.39

angle of attack increases as r decreases—it reaches 18.4° when $r = 0.78$ meters (just above the cowling). This leads to a (slightly) negative torque. To avoid this unfavorable situation, one usually introduces a twist in the airfoil to reduce the angle of attack at small radii. However, this would destroy the symmetry and the turbine would operate differently depending on the direction of the air flow. It turns out, that this particular turbine configuration, known as the **Wells turbine** is designed especially for situations in which the airflow changes alternatively its direction. See the WAVEGEN project in Chapter 6. The specific design suggested in this problem is not optimal. One should either use a larger cowling or a somewhat different symmetric air foil.

The delivered power should be limited to

$$P_A = \frac{16}{27}\tfrac{1}{2}\rho V^3 A_v = \frac{16}{27}\tfrac{1}{2} \times 3.87 \times 28.6^3 \times \pi\frac{D^2}{4} = 142 \quad \text{kW}. \tag{19}$$

A problem arises if one considers that the number of turbine blades is a design choice and can arbitrarily be increased (until the solidity becomes 1). Thus, the power generated can be much higher than estimated here. What happens, of course, is that we neglected the interference of one blade with the next.



Solution of Problem 15.39

Prob 15.40 Compare two different sites for wind farms: in one the wind blows at a steady 15 m/s, all the time; in the other the wind blows at a steady 20 m/s. exactly half of the time. during the other half, there is no wind. Both are at sea level.

a. What is, in the two sites, the yearly amount of electrical energy generated by a propeller type wind turbine having a rotor diameter of 50 m and an overall efficiency of 60%?

....

The power delivered by the turbine is,

$$\begin{aligned} P_D =& \frac{16}{27}\frac{1}{2}\rho < v >^3 A\eta = \frac{16}{27}\frac{1}{2}1.29 < v >^3 \times \frac{\pi}{4} \times 50^2 \times 0.6 \\ =& 450 < v >^3 \quad \text{W}. \end{aligned} \tag{1}$$

In Site 1, v is constant, hence, $\overline{v} = < v > = 15$ m/s, and $< v >^3 = 3{,}380$.

In Site 2, $v = 20$ m/s half the time and $v = 0$ the rest of the time or, $v^3 = 8{,}000$ half the time, i.e. $< v >^3 = 4{,}000$.

Consequently,

$$\begin{cases} P_{D_1} = 1.52 \text{ MW}, \\ P_{D_2} = 1.80 \text{ MW}. \end{cases} \tag{2}$$

The electric energy generated in one year is,

$$\begin{cases} W_{E_{year_1}} = 3.16 \times 10^7 \times 1.52 \times 10^6 = 48 \times 10^{12} \quad \text{J} \quad \text{or} \quad 13.3 \times 10^6 \quad \text{kWh}, \\ W_{E_{year_2}} = 3.16 \times 10^7 \times 1.80 \times 10^6 = 57 \times 10^{12} \quad \text{J} \quad \text{or} \quad 15.8 \times 10^6 \quad \text{kWh}. \end{cases} \tag{3}$$

Site 1, the one with the steady wind, generates 48 TJ or 13.3 million kWh per year, while Site 2, generates 57 TJ or 15.8 million kWh per year.

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b. If the plants, above, were installed in Cochabamba, 2558 m high in the Bolivian Andes, what would be their estimated yearly production of electricity?

....

Assuming a scale height for Earth's atmosphere of 8800 m, the atmospheric density at Cochabamba is

$$\rho_{Cochabamba}/\rho_{Sea\ level} = \exp -\frac{h_{Cochabamba}}{8800} = \exp -\frac{2558}{8800} = 0.75. \tag{4}$$

Since the power of a wind turbine depends linearly on the density of the medium, the yearly electricity production at Cochabamba will be 75% of that at sea level.

Site 1 would produce 36 TJ or 10 milion kWh, while Site 2 would produce 43 TJ or 12 million kWh every year.

Solution of Problem 15.40

Prob 15.41 A wind turbine with 30 m radius, and 60% efficiency is installed at a sea level site in which the average wind regimen is given in the table below:

	Wind (m/s)	% of time
1	0	10
2	5	25
3	10	35
4	15	25
5	20	5

Assume that the wind velocity in each wind slot (1 though 5) is the cubic mean in that slot.

The cost of the generator is \$0.25 dollar per rated watt and that of the turbine is \$900/m^2 of swept area. The yearly cost during the lifetime of the plant is 15% of the cost of the plant for both the generator and the wind turbine. Assume no other costs.

You can chose a generator of a rated power equal to the power the turbine delivers when the wind is 5 m/s. This means that if the wind speed exceeds 5 m/s, the plant will only generate the power corresponding to a 5 m/s wind.

Another possibility is the choice of a generator rated at the power corresponding to 10 m/s.

You can also chose a generator rated at 15, or at 20 m/s.

You have four different choices. Clearly, the larger the generator, the more expensive the plant.

a. Calculate the cost of the kWh generated for each of the above choices. If you aim to produce the cheapest electricity, what size generator would you pick?

....

The power delivered by the turbine/generator is

$$P_D = \frac{16}{27}\frac{1}{2}\rho A\eta <v>^3 = \frac{16}{27}\frac{1}{2} \times 1.29 \times \pi \times 30^2 \times 0.6 <v>^3 = 648 <v>^3. \quad (1)$$

If the generator is rated to be at full capacity when the the wind is 5 m/s, then it must be rated at, $P_{g_{rated}} = 648 <5>^3 = 81{,}000$ W, and so on for each slot. The cost of the generator, which is $C_{gen} = 0.25 P_{g_{rated}}$, can be calculated and we get the total cost of the plant, $C_{plant} = C_{turb} + C_{gen}$. $C_{turb} = 900 \times \pi r^2 = \$2,545,000$. The yearly cost is $c_{year} = 0.15 C_{plant}$.

We must now calculate the overall value of $<v>^3$ for each slot, that is, we must find the weighted cubic mean wind velocity corresponding to each choice of rated generator power:

Solution of Problem 15.41

$$\begin{cases} 1 & <v>^3 = (0.25+0.35+0.25+0.05)\times 5^3 = 112.5 \\ 2 & <v>^3 = 0.25\times 5^3 + (0.35+0.25+0.05)\times 10^3 = 681.3 \\ 3 & <v>^3 = 0.25\times 5^3 + 0.35\times 10^3 + (0.25+0.05)\times 15^3 = 1394 \\ 4 & <v>^3 = 0.25\times 5^3 + 0.35\times 10^3 + 0.25\times 15^3 + 0.05\times 20^3 = 1625 \end{cases}$$

The average generated power, $< P_g >= 648 < v >^3$, is, of course, much smaller than the rated power, $P_{g_{rated}}$. The total yearly energy production is $W_{year} = 8760 < P_g >$. Finally, the cost of electricity is $c = c_{\frac{year}{W_{year}}}$

Collecting our results,

Wind speed slot (m/s)	5	10	15	20
Cubic wind speed (m^3/s^3)	112.5	681.3	1,394	1,625
Rated generator power (kW)	81	648	2,187	5,184
Cost of generator ($)	20,250	162,000	546,750	1,296,000
Cost of plant ($)	2,565,250	2,707,000	3,091,750	3,841,000
Yearly cost ($)	384,787	406,050	463,762	576,150
Average generator power (kW)	72.9	441	903	1,053
Deliverable yearly energy (kWh)	639,900	3,875,234	7,929,072	9,243,000
Cost of electricity ($/kWh)	0.601	0.1047	**0.0584**	0.0623

It can be seen that the cheapest electricity is generated when the generator is rated at the output corresponding to a wind velocity of 15 m/s.

The cheapest electricity costs 5.84 cents per kWh.

---

b. For the best choice of Item a, what is the total cost of the facility in dollars per rated kW?

....

For Choice 4, the plant costs is 3.092 million dollars and the generator is rated at 2,188 kW, a ratio of 3,092,000/2,188 = 1413 dollar per kW.

The facility costs $1400 per rated kW.

Solution of Problem 15.41

Prob 15.42 A matter accelerator is mounted at the edge of a mesa and launches a spherical projectile horizontally towards a level plain which is 100 m below the launching device. The problem is to determine how far the projectile goes before impacting the ground. First, assume that there is no air so that there is no aerodynamic drag. This will set the maximum value of the range (distance from launcher to impact point.) Next consider the case when there is air at STP, however, to simplify the problem, assume that the air drag influences only the horizontal component of the motion of the projectile, not the vertical component of the motion

The necessary data to solve the problem include:

- a **Initial velocity, v_0, is 720 km/hr.**
- b **The spherical projectile does not spin.**
- c **The projectile is hollow and is made of iron 2 cm thick.**
- d **The outer diameter of the projectile is 1 m.**
- e **The density of iron is 7874 kg/m^3.**
- f **Assume that the drag coefficient, C_D, is 1.0.**

....

1 – <u>Airless Solution.</u>

The time of flight of the projectile is determined by the length of time, t_v, it takes for it to fall the 100 meters from the level of the cannon to that of the plain.

$$h = \tfrac{1}{2} g t^2, \tag{1}$$

$$100 = \tfrac{1}{2} \times 9.81 \times t^2, \tag{2}$$

$$t = 4.51 \quad s. \tag{3}$$

Since there is no air drag, the horizontal component of the velocity of the projectile is constant. The projectile mantains its 200 m/s horizontal velocity until the moment of impact. Hence it will reach a distance of

$$d = 200 \times 4.51 = 902 \quad \text{m}. \tag{4}$$

The impact point is 902 m from the launch point.

2 – <u>Solution with air.</u>

We need to calculate the mass of the projectile.

The volume of a sphere with a radius, r, is $\frac{4}{3}\pi r^3$. The volume of the space between two concentric spheres (radii r_1 and r_2) is

$$V = \frac{4}{3}\pi(r_1^3 - r_2^3) = \frac{4}{3}\pi(0.5^3 - 0.48^3) = 0.06035 \quad \text{m}^3. \tag{5}$$

The corresponding mass is

$$m = 7874 \times 0.6035 = 475.2 \quad \text{kg}. \tag{6}.$$

Solution of Problem 15.42

The problem statement specifies that air drag does not affect the time of fall (in reality it does). Thus, the problem is to determine how far does the projectile go in 4.51 seconds when it travels in air at STP.

The force the air exerts on the projectile is

$$F = -a_2 v^2 = -\tfrac{1}{2}\rho C_D A = -\tfrac{1}{2}1.29 \times 1 \times \pi \times 0.5^2 v^2 = -0.5066 v^2. \quad (7)$$

The resulting acceleration is

$$\frac{dv}{dt} = \frac{F}{m} = -\frac{0.5066 v^2}{475.2} = -0.001066 v^2. \quad (8)$$

Separating variable, one obtains the easily solved differential equation,

$$\frac{dv}{v^2} = -0.001066 dt, \quad (9)$$

which yields,

$$0.001066 t = \frac{1}{v} - \frac{1}{v_0}, \quad (9)$$

from which

$$v = \frac{1}{\frac{1}{v_0} + 0.001066 t} = \frac{1}{0.005 + 0.001066 t} \quad (10)$$

The last step follows from $v_0 = 200$ m/s.

The distance, d, the projectile will fly is

$$d = \int_0^t v dt = \int_0^t \frac{dt}{0.005 + 0.001066 t} = \left.\frac{\ln(0.005 + 0.001066 t)}{0.001066}\right|_0^{4.51} = 632 \quad \text{m.} \quad (11)$$

The impact point is 632 m from the launch point.

Solution of Problem 15.42

Prob 15.43 You are in charge of a sea level wind farm that has bought a number of horizontal-axis propeller-type wind turbines having the following specifications:

Cut-in wind speed:	**3.5 m/s.**
Cut-out wind speed:	**27 m/s.**
Rotor:	**3-blade, 104-m diameter.**
Efficiency of the turbine/generator system (referred to the available wind power):	**50%.**

A generator of selected power and the corresponding gear box can be matched to the turbine.

The turbine plus tower cost \$2.5 million. The generator cost depends, of course, on the chosen power: it is \$278 per kW of electric output. The cost of money is 12%/year. Operating cost per unit (1 turbine plus generator) is \$230,000.00/year. This includes insurance, taxes, and salaries.

All you know about the wind regime at the selected site is that it has, traditionally, been 12 m/s on average. This is obviously a very good site for wind power generation.

There will be times of the year when there will be enough wind to drive the generator to full power. Develop a formula that yields the total amount of energy (in kWh) delivered in one full year by the generator counting only the occasions when the generator is developing its full rated power, (P_g). Disregard the power generated at less than full rated power.

a. What is the cost of the electricity above, in mills/kWh† (use 3 significant figures, and, as instructed above, disregard the energy generated at less than full power) when the rated generator power is $P_g = 3.0$ MW. Repeat for, $P_g = 3.6$ MW.

....

The power delivered by the system is

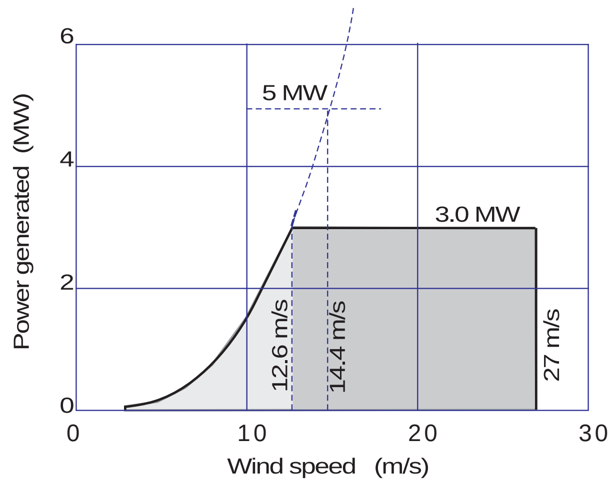
$$P_D = \frac{16}{27}\tfrac{1}{2}\rho v^3 A\eta = \frac{16}{27} \times \tfrac{1}{2} \times 1.2 \times \frac{\pi}{4} \times 104^2 \times 0.5 v^3 = 1510 v^3. \qquad (1)$$

We nee to know the wind speed, v_{fp}, that delivers the full power, P_g. We want the system to deliver either 3 or 3.6 MW. This corresponds to 2 different values of velocity:

$$v_{fp} = \left(\frac{P_g}{1510}\right)^{1/3} = \begin{cases} v_{fp} = 12.6 \text{ m/s} & \text{for } P_D = 3.0 \text{ MW}, \\ v_{fp} = 13.4 \text{ m/s} & \text{for } P_D = 3.6 \text{ MW} \end{cases} \qquad (2)$$

† Mill is one tenth of a cent.

Solution of Problem 15.43



The figure, above, depicts the characteristics of the system (for the 3 MW case). The darker shaded rectangle is the region of operation in which we are interested. We have to determine how many hours per year the wind velocity is above v_{fp} (12.6 or 13.4 m/s, depending on the choice of generator) and less than $v_{cutt-off}$ (27 m/s). For lack of better information we have to assume that the wind obeys a Rayleigh distribution. The question is what is the probability, $\Pi(v)$, of the wind blowing at speeds above v_{fp}.

$$\Pi(v_{fp}) = \exp\left(-\frac{\pi}{4}\frac{v_{fp}^2}{12^2}\right) = \exp\left(-\frac{\pi}{4}\frac{\left(\frac{P_g}{1510}\right)^{2/3}}{12^2}\right) = \exp\left(-4.13 \times 10^{-5} P_g^{2/3}.\right) \tag{3}$$

However, if the wind speed exceeds 27 m/s, the system shuts down. The probability of this occurring is

$$\Pi(v_{sd}) = \exp\left(-\frac{\pi}{4}\frac{v_{sd}^2}{12^2}\right) = \exp\left(-\frac{\pi}{4}\frac{27^2}{12^2}\right) = 0.019. \tag{4}$$

So, the probability of the generator actually delivering full power is

$$\Pi(v_{FP}) = \exp\left(-4.13 \times 10^{-5} P_g^{2/3}\right) - 0.019. \tag{5}$$

The energy generated in a year is

$$W_{FP} = 8760 \times \Pi(v_{FP}) P_g = 8760 \left[\exp\left(-4.13 \times 10^{-5} P_g^{2/3}\right) - 0.019\right] P_g, \tag{6}$$

where 8760 is the number of hours in a year, and W_{FP} is in Wh/yr. For the two selected values of P_g, the energy produced is

$$\begin{cases} \text{For } P_g = 3.0 \text{ MW}, W_{FP} = 1.06 \times 10^{10} \text{ Wh/yr}, \\ \text{For } P_g = 3.6 \text{ MW}, W_{FP} = 1.14 \times 10^{10} \text{ Wh/yr}. \end{cases} \tag{7}$$

Solution of Problem 15.43

The yearly costs (in dollars) are $230,000 (operating cost) plus $0.12 \times 2.5 \times 10^6$ (turbine cost) plus $0.12 \times 0.278 \times P_g$ (generator cost). This amounts to $530,000 + 0.0334 P_g$, or

$$\begin{cases} \$630,000, \\ \$650,000. \end{cases} \quad \text{per year.} \tag{8}$$

The corresponding cost per kWh is

$$\begin{cases} \dfrac{\$630,000}{1.06 \times 10^7} = 0.0594, \\ \dfrac{\$650,000}{1.14 \times 10^7} = 0.0570. \end{cases} \quad \$/\text{kWh}. \tag{9}$$

The cost of electricity is 59.4 mills/kWh if the generator is rated or 57.0 mills/kWh if the generator is rated at 3.6 MW.

Do not write a dissertation! Answers for the questions below should be terse. In all cases the blade length (radius) is the same.

b. Make a very rough guess: what is the torque the propeller exerts on the shaft that leads to the input of the gear box (just a ballpark figure). Assume the machine is operating at the rated wind speed, the one that just delivers full power. Make plausible assumptions. Consider the 3.6 MW case.

....

A large machine of the type described above may operate at a tip speed ratio of, say, 7. That means that the angular velocity is expected to be

$$\omega = \frac{v_1 \lambda}{R}. \tag{10}$$

From Equation 2, for the 3.6 MW case,

$$v = \left(\frac{3.6 \times 10^6}{1510} \right) = 13.4 \quad \text{m/s}, \tag{11}$$

$$\omega = \frac{13.4 \times 7}{52} = 1.8 \quad \text{rad/sec}. \tag{12}$$

To deliver 3.6 MW at this angular velocity, the torque must be,

$$\Upsilon = \frac{P_g}{\omega} = \frac{3.6 \times 10^6}{1.8} = 2 \times 10^6 \quad \text{newton meters}. \tag{13}$$

The wind turbine will deliver some 2 million newton meters.

Solution of Problem 15.43

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c. What would happen (qualitatively) to the torque if the turbine had only 2 blades instead of 3, that is, if the solidity were decreased?

....

With lower solidity the torque would decrease and the angular velocity would increases.

To first order, the performance will be about the same.

---

d. Which of the two turbines in the question above would, presumably, have less wake rotation loss?

....

The 3-blade machine would have higher torque hence it would have somewhat higher wake losses.

Solution of Problem 15.43

Prob 15.44 An object masses 10 kg, has a frontal area of 0.3 m^2, and a drag coefficient of 1.1. This object is dropped from an airplane flying at 5000 m (initial vertical velocity is zero). The horizontal velocity plays no role—it can be taken as zero, also. The atmospheric density (in kg/m^3) is a function of height and is given by

$$\rho = 1.29 \exp\left(-\frac{h}{8000}\right). \tag{1}$$

The height, h, is in meters above sea level. Assume that the acceleration of gravity, g, is height independent. Describe quantitatively what happens. Calculate the maximum vertical velocity acquired by the object.

This will lead to a differential equation. Do not attempt to solve it analytically. Use a numerical solution with a time step of 1 second. Few iterations will be needed and, notwithstanding the very coarse time steps, will yield a good estimate of the velocity. Surprisingly, you will overestimate the correct velocity by less than 0.5%.

The object will accelerate downward under the influence of gravitaty. As the speed increases the air drag will increase fast . When the drag force equals the weight of the object, the object will have reached its maximum vertical speed. From here on, the speed will decrease owing to the growing density of the air.

The weight of the object is

$$F_g = mg = 10 \times 9.81 = 98.1 \quad \text{N}. \tag{2}$$

The air drag force on the object is

$$F_D = \tfrac{1}{2}\rho v^2 A C_D = 0.165\rho v^2 \tag{3}$$

where

$$\rho = 1.29 \exp\left(-\frac{h}{8000}\right). \tag{4}$$

The net downward acceleration is

$$\gamma = \frac{F_g - F_D}{m} = \frac{98.1 - 0.165\rho v^2}{10} = 9.81 - 0.0165\rho v^2. \tag{5}$$

At time zero we have :
$t_0 = 0$
$v_0 = 0$
$h_0 = 5000$

At time step "i":

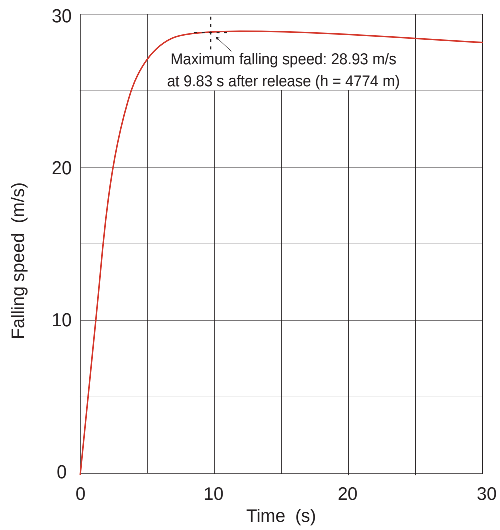
Solution of Problem 15.44

$t_i = t_{i-1} + \Delta T$
$v_i = v_{i-1} + \gamma_{i-1} \times \Delta t$
$h_i = h_{i-1} - v_i \times \Delta t$
$\rho_i = 1.29 \times \exp -\frac{h_i}{8000}$
$F_{D_i} = 0.165 \times \rho_i \times v_i^2$
$F_i = 98.1 - F_{D_i}$
$\gamma_i = \frac{F_1}{m}$

t	v	h	ρ	F_D	F	γ
(s)	(m/s)	(m)	kg/m^3	N	N	m/s^2
0	0	5000	0.6905	0	98.1	9.81
1	9.81	4990	0.691	10.98	87.12	8.71
2	18.52	4972	0.693	39.22	58.87	5.89
3	24.41	4947	0.695	68.33	29.77	2.98
4	27.39	4920	0.607	86.31	11.79	1.18
5	28,57	4891	0.700	94.24	3.86	0.39
6	28.95	4862	0.702	97.15	0.95	0.09
7	29.05	4833	0.705	98.15	-0.05	-0.005

From the coarse numerical solution, we find a maximum speed of 29.05 m/s.

Below: exact solution. It is interesting to observe that the coarse solution yields almost the exact maximum velocity but underestimates both the time necessary to reach this velocity (7 s versus 9.83 s) and how far the body falls before reaching the maximum velocity (187 m versus 226 m).



Solution of Problem 15.44

Prob 15.45 The GE 2.5-xl wind turbine has the following characteristics:
$P_{rated} = 2.5$ **MW.**
$v_{cutin} = 3.5$ **m/s.**
$v_{rated} = 12.5$ **m/s.**
$v_{cutout} = 25$ **m/s.**
Three 50-m rotor blades.
What is the rotor loading when the wind velocity is 10 m/s? A new generator is used having a rated power equal to the power the turbine delivers when there is a 10 m/s wind.

....

The swept area of the turbine is,

$$A_v = \pi r^2 = 3.1415 \times 50^2 = 7854 \quad \text{m}^2. \tag{1}$$

When $v_{cutin} < v < v_{rated}$, the power generated, P_g is proportional to v^3,

$$P_g = \Lambda v^3, \tag{2}$$

$$\Lambda = \frac{P_{g_{rated}}}{v^3_{rated}} = \frac{2.5 \times 10^6}{12.5^3} = 1280. \tag{3}$$

At 10 m/s,

$$P_g = 1280 \times 10^3 = 1.28 \times 10^6 \quad \text{W}. \tag{4}$$

The rotor loading is

$$RL = \frac{1.28 \times 19^6}{7854} = 163 \quad \text{W/m}^2. \tag{5}$$

The rotor loading is 163 W/m^2.

Solution of Problem 15.45

Prob 15.46 In Southern California, there are locations in which the Santa Ana winds blow steadily at 100 km/h during 10% of the year. During the rest of the time there is negligible wind. You want to adapt the GE 2.5-xl to these conditions. You keep the same generator but change the gear box and the rotor (still 3 blades). Assuming all efficiencies are still the same, what is the length of the rotor blade?

....

The wind velocity (100 km/h) is 27.8 m/s.

The power generated is proportional to $A_v v^3$, and is still 2.5 MW,

$$A_{v_1} v^3_{rated_1} = A_{v_2} v^3_{rated_2}, \tag{1}$$

$$A_{v_2} = \frac{v^3_{rated_1}}{v^3_{rated_2}} A_{v_1} = \frac{12.5^3}{27.8^3} \times 7854 = 714 \quad \mathrm{m}^2. \tag{2}$$

$$r = \sqrt{\frac{714}{3.1416}} = 15.1 \quad \mathrm{m}. \tag{3}$$

All it takes is tiny (and inexpensive) 15.1-m blades.

Solution of Problem 15.46

Prob 15.47 Assuming optimum adjustment, what is the velocity of the wind just behind the rotor disk of the turbine in Problem 15.46?

....

The "retarded" wind just behind the rotor is 2/3 of the undisturbed wind, i.e., it is 18.5 m/s.

Solution of Problem 15.47

Prob 15.48 Estimate the force the wind exerts on the rotor disk of the turbine in Problem 15.46. We are not talking about the torque force that turns the rotor, but rather, the pushing force that tries to topple the tower.

....

The pressure drop across the rotor disk is (see the Rankine-Foude theorem),

$$\Delta p = \tfrac{1}{2}\rho(v_1^2 - v_3^3), \tag{1}$$

but, $v_3 = v_1/3$, hence,

$$\Delta p = \tfrac{1}{2}\rho \times \frac{8}{9}v_1^2 = \tfrac{1}{2} \times 1.2 \times \frac{8}{9} \times 27.8^2 = 412 \quad \text{Pa.} \tag{2}$$

Since the swept area is (Prob 13) 714 m^2, the force is

$$F = \Delta p A_v = 412 \times 714 = 294,000 \quad \text{N.} \tag{3}$$

The wind exerts a force of 294 kN on the swept area of the turbine.

****

Another way of solving this problem:

$$P_g = v_2 F\eta. \tag{4}$$

We have to find the efficiency, η, of the turbine.
We know it generates 2.5 MW when driven by a 27.78 m/s wind:

$$2.5 \times 10^6 = \frac{16}{27} \times \tfrac{1}{2}\rho \times 27.78^3 \times 714\eta = 5.44 \times 10^6\eta, \tag{5}$$

from which, $\eta = 0.46$.

From equation, 4,

$$F = \frac{P_g}{v_2\eta} = \frac{2.5 \times 10^6}{18.5 \times 0.46} = 294{,}000 \quad \text{N.} \tag{5}$$

****

However, it is not a good idea to apply the drag equation, $p = \frac{1}{2}\rho v^2 A_v C_D$, because we have no clue as to what C_D is. In this case it is 2.1—larger than that of a flat plate (1.28).

Solution of Problem 15.48

Prob 15.49 A boat drifts downwind with a constant velocity, W, relative to the water. It is equipped with a large "sail" which happens to be a big rectangular wing, 4 m heigh and 2 m in chord. The airfoil of this wing is NACA 4412. For this downwind motion the airfoil is irrelevant because the wing is set perpendicular to the wind (and boat) direction and acts as a flat surface with a coefficient of drag, $C_D = 1.2$. Assume that the rest of the boat (except the 4 m by 2 m "sail'") does not interact with the wind. The force of the water resisting the forward motion of the boar is $F_W = 10W^2$. The boat covers the distance A to B, a total of 2 km, in 20 minutes. Use an air density of 1.22 kg per m^3.

a - What is the wimd velocity?

....

The boat moves directly downwind with a velocity, W, hence the wind velocity relative to the boat is

$$U = V - W \tag{1}$$

where V is the wind velocity, and U is the induced velocity acting on the sail.

This induced wind exerts a force, F_U on the boat, while the water resistance exerts a retarding force, F_W. Since the boat is moving at constant speed, the two forces must equal one another.

$$F_U = \frac{1}{2}\rho(V - W)^2 A C_D = 10W^2 = F_W, \tag{2}$$

$$\frac{1}{2}1.22(V - W)^2 \times 8 \times 1.2 = 10W^2, \tag{3}$$

$$0.5856(V - W)^2 - W^2 = 0. \tag{4}$$

$$0.7652(V - W) = W, \tag{5}$$

$$V = 2.307W. \tag{6}$$

The boat velocity is

$$W = \frac{2000}{20 \times 60} = 1.667 \quad \text{m/s.} \tag{7}$$

$$V = 2.307 \times 1.667 = 3.845 \quad \text{m/s.} \tag{8}$$

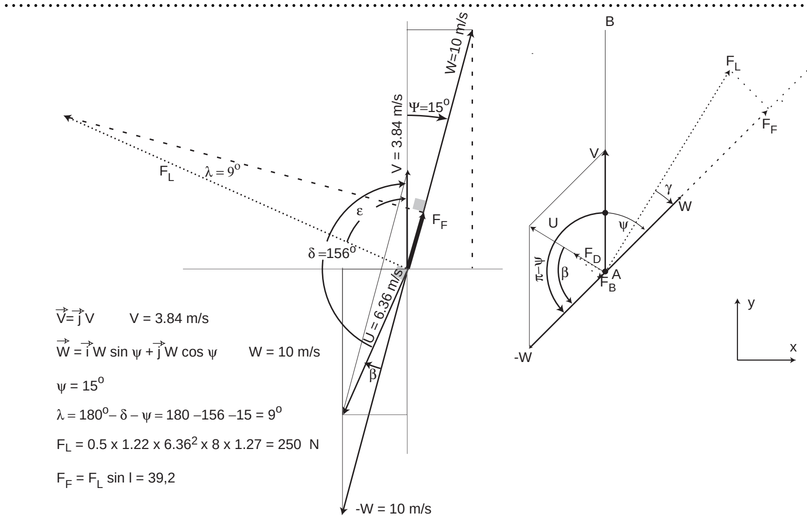
The wind velocity, V, is 3.84 m/s.

---

b - Now assume that the same boat, under the same wind, starts from A but heads at an angle, ψ, to the starboard (right) of the AB line. It travels in this direction until it reaches Point C, equidistant from A and B. Next it heads from this point directly to A. What is

the value of ψ that minimizes the time for the ACB route? What is the minimum time for the ACB trip? Always adjust the setup angle of the airfoil so that the angle of attack is 8°. The boat can only move forward because a large keel resists any sidewards drift.

....



If the boat moves with a velocity, W, in the direction that makes an angle, ψ, with the AB direction, it will experience a wind of magnitude, $-\vec{W}$, owing to this motion alone. Combined with the wind velocity, $\vec{V}$, the induced wind perceived by the boat will be

$$\vec{U} = \vec{V} - \vec{W}. \tag{9}$$

$$\vec{V} = \vec{j}V. \tag{10}$$

$$-\vec{W} = -\vec{i}W\sin\psi - \vec{j}W\cos\psi. \tag{11}$$

$$\vec{U} = -\vec{i}W\sin\psi + \vec{j}(V - W\cos\psi). \tag{12}$$

$$U = V\sqrt{1 + \left(\frac{W}{V}\right)^2 - 2\left(\frac{W}{V}\right)\cos\ \psi} \equiv V\Gamma. \tag{13}$$

$$\Gamma \equiv \sqrt{1 + \left(\frac{W}{V}\right)^2 - 2\left(\frac{W}{V}\right)\cos\ \psi} = \sqrt{1 + 0.0678W^2 - 0.521W\cos\psi}. \tag{14}$$

$$\vec{U}\cdot\vec{V} = UV\cos\beta = -W^2 + vW\cos\psi. \tag{15}$$

$$\cos\beta = \frac{W/V + \cos\psi}{\Gamma} = \frac{0.260W + \cos\psi}{\sqrt{1 + 0.0678W^2 - 0.521W\cos\psi}}. \tag{16}$$

Solution of Problem 15.49

The induced wind, $\vec{U}$, will generate a lift force, F_L, (with forward a component, F_F, along the direction of motion of the boat) and a drag force, F_D, (with at retarding component, F_B).

$$\begin{aligned}F_L &= \tfrac{1}{2}\rho U^2 C_L A = \tfrac{1}{2} \times 1.22 \times 8 \times 1.27 U^2 = 6.20\ U^2 = 6.20\ V^2\Gamma^2 \\ &= 6.20\ V^2 \times \left[1 + \left(\frac{W}{V}\right)^2 - 2\left(\frac{W}{V}\right)\cos\ \psi\right] \\ &= 6.20 \times 3.84^2 \times \left[1 + 0.0678W^2 - 0.521W\cos\psi\right] \\ &= 91.4 + 6.20W^2 - 47.6W\cos\psi. \end{aligned} \tag{17}$$

We used $C_L = 1.27$ which is the value of C_L for the NACA 4412 airfoil when the angle of attack is 8°, as prescribed by the problem statement.

The component of lift along the direction of motion of the boat is

$$\begin{aligned}F_F &= F_L\cos\gamma = F_L\sin\beta = F_L\sin\left[\arccos\left(\frac{W/V + \cos\psi}{\Gamma}\right)\right] \\ &= \left(91.4 + 6.20W^2 - 47.6W\cos\psi\right)\sin\left[\arccos\left(\frac{0.260W + \cos\psi}{\sqrt{1 + 0.0678W^2 - 0.521W\cos\psi}}\right)\right]\end{aligned} \tag{18}$$

Let $\Lambda \equiv \frac{C_D}{CL} = \frac{0.0106}{1.27} = 0.0083$. Under all conditions,

$$F_D = 0.0083F_L. \tag{19}$$

$$F_B = F_D\cos\beta = 0.0083F_L \times \frac{0.260W + \cos\psi}{\sqrt{1 + 0.0678W^2 - 0.521W\cos\psi}}. \tag{20}$$

The force of the wind acting on the boat in the direction of its motion is

$$F_{wind} = F_F - F_B = 10W^2. \tag{21}$$

Solution of Problem 15.49

Prob 15.50 The only information you have about a proposed site for a wind farm is that the wind blows at speeds above 25 m/s 90 hours per year. What is the most probable average wind velocity, $\overline{v}$?

....

90 hours per year corresponds to a fraction, 90/8760=0.0103 of the year. In absence of any further information, we are reduced to assume that the wind at the sight obeys a Rayleigh velocity probability distribution. If so, the complementary cumulative distribution function is

$$\exp\left[-\frac{\pi}{4}\left(\frac{25}{\overline{v}}\right)^2\right] = 0.0103, \tag{1}$$

$$-\frac{\pi}{4}\left(\frac{25}{\overline{v}}\right)^2 = \ln 0.0103 = -4.58, \tag{2}$$

$$\overline{v} = 10.4 \quad \text{m/s}. \tag{3}$$

The most probable average wind velocity is 10.4 m/s

Solution of Problem 15.50

Prob 15.51 What is the least wind velocity that will cause a wind turbine having a propeller of 50 m radius to produce 3 MW of electric power? The efficiency of the turbine is 52% and the electric generator is assumed to be 100% efficient. The turbine is at sea level. .... The power generated by a wind turbine is given buy

$$P_g = \frac{16}{27}\tfrac{1}{2}\rho v^3 \pi r^2 \eta = \frac{16}{27}\times\tfrac{1}{2}\times 1.22 v^3 \times \pi \times 50^2 \times 0.52 = 1476 \times v^3 = 3\times 10^6. \quad (1)$$

from which

$$v = 12.7 \quad \text{m/s}. \quad (2)$$

The minimum wind velocity is 12.7 m/s.

Solution of Problem 15.51