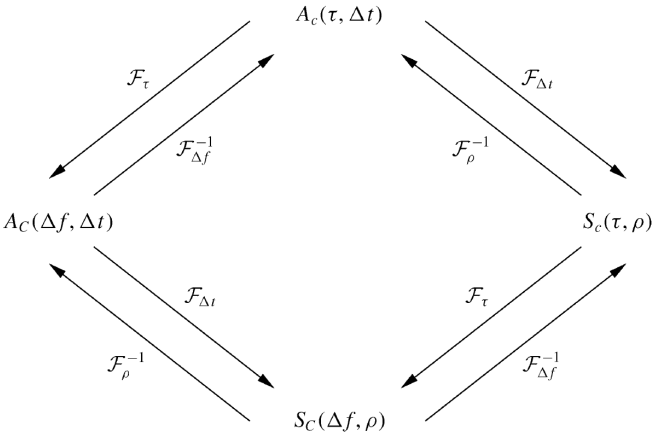
EE359 Discussion Session 3
Capacity of Flat and Frequency Selective Channels

February 5, 2020

Note about scattering functions



For deterministic response

$$h(t,\tau) = \sum_i \alpha_i(t) e^{-j2\pi(t-\tau_i(t))} \delta(t - \tau_i(t))$$

With no movement (Doppler), $h(t,\tau) \to h(\tau)$ becomes time-invariant response (LTI system)

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t (Time) $\longleftrightarrow$ ρ (Doppler)

τ (Delay) $\longleftrightarrow$ f (Frequency)

What is Capacity?

- Maximum achievable rate with *no* errors
- Maximum mutual information over all input distributions
- Usually found by proving matching lower (achievability) and upper (converse) bounds
- Often easy to bound, but hard to prove

Notions of Capacity in Wireless Systems

- AWGN Capacity
- Only CSI distribution known at TX and RX
- CSI at RX only
  - With or without outage
- CSI at TX and RX
  - With or without adaptation
  - With or without outage

AWGN Capacity

Capacity of fixed channel with **no fading**.

$$C = B \log_2(1 + \gamma)$$

Can be used to bound other settings

CSI Distribution Known

Really complicated

Channel capacity with CSIR

Two notions of capacity

Ergodic capacity

$$C = B \int \log_2 (1 + \gamma)\, p(\gamma) d\gamma$$

- Achieved by coding over fading states γ

Channel capacity with CSIR

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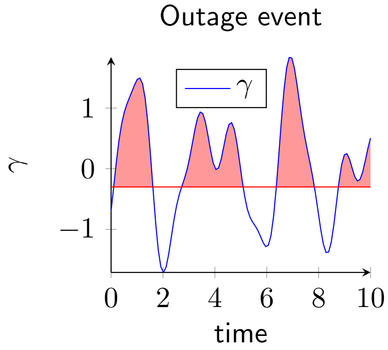
Outage capacity

- Find out minimum SNR γ_{min} needed to achieve outage prob P_{out}. If it violates power constraint then $C = 0$
- Transmit at that SNR thereby achieving

$$C = (1 - P_{out}) B \log_2(1 + \gamma_{min})$$

Outage capacity (without CSIT)



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Capacity with CSIR and CSIT

System model

$$y[i] = \sqrt{g[i]}x[i] + n[i]$$
$$\sqrt{g[i]} \sim \text{fading distribution}$$
$$E[|x[i]|^2] \leq \bar{P}$$

- $g[i]$ known at transmitter and receiver

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- What is capacity with fixed TX power?

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- $g[i]$ known at transmitter and receiver

- What is capacity with fixed TX power?
- Can capacity increase with rate and power adaptation?

Towards optimally exploiting the CSI $g[i]$

Idea

Vary transmit power P and rate R as a function of $g[i]$ or equivalently, of $\gamma = \frac{g[i]\bar{P}}{N_0 B}$

- Power $P(\gamma)$
- Rate $R = B \log\left(1 + \gamma \frac{P(\gamma)}{\bar{P}}\right)$

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Notation

- Fading distribution given by $p(\gamma)$
- Assuming that Tx uses *fixed* average power $\bar{P}$.

Optimization problems for optimal CSIT and CSIR use

Optimization problem

$$\max_{P(\gamma)} \int B \log\left(1 + \frac{P(\gamma)}{\bar{P}}\gamma\right) p(\gamma) d\gamma$$
$$\text{s.t. } E[P(\gamma)] \leq \bar{P}$$
$$P(\gamma) \geq 0 \ \forall \ \gamma$$

Optimization problem (discrete γ)

$$\max_{\mathbf{P}(\gamma)} \sum_i B \log\left(1 + \frac{P(\gamma_i)}{\bar{P}}\gamma_i\right) p(\gamma_i)$$
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Optimization problem (discrete γ)

$$\min_{\mathbf{P}(\gamma)} \sum_i -B \log\left(1 + \frac{P(\gamma_i)}{\bar{P}}\gamma_i\right) p(\gamma_i)$$
$$\text{s.t. } E[P(\gamma)] \leq \bar{P}$$
$$P(\gamma_i) \geq 0 \ \forall \ i$$

Lagrangian Methods

Optimization problem

$$\min_{x \in \mathcal{S}} f_0(x),$$
$$\text{s.t. } f_i(x) \leq 0, \ \forall i = 0, \ldots, m,$$
$$\text{s.t. } h_i(x) = 0, \ \forall i = 0, \ldots, p,$$

Lagrangian

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

- ν, λ are called *Lagrange multipliers* or *dual variables*
- Process sometimes refered to as "regularlizing constraints".

Lagrangian Duality

Dual problem

$$g(\lambda,\nu) = \inf_{x\in\mathcal{S}} L(x,\nu,\lambda)$$
$$= \inf_{x\in\mathcal{S}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

Solving the dual problem

- Find closed form expression for $g(\lambda,\nu)$: $\nabla_x L(x,\lambda,\nu) = 0$
- $g(\lambda,\nu)$ is now a convex optimization problem (in our case of *one variable*)!

Why consider dual problem?

- Let $p^\star$ be the optimal value of the original optimization problem:

$$f_0(\tilde{x}) = p^\star, \tilde{x} \text{ is feasible.}$$

- Let $d^\star$ be the optimal value of the dual problem:

$$g(\tilde{\lambda}, \tilde{\nu}) = d^\star$$

Lower Bound Property

$p^\star \geq d^\star$ as long as $\lambda \geq 0$.

Strong Duality

$p^\star = d^\star$ for many problems! **True in our case**

Take EE364a/b to understand when and why this is true

Example: Least squares

$$\min x^\intercal x$$
$$\text{s.t } Ax = b$$

Dual Function

$$L(x, v) = x^\intercal x + \nu^\intercal (Ax - b)$$
$$\nabla_x L(x, \nu) = 2x + A^\intercal \nu = 0 \Rightarrow x = -(1/2)A^\intercal \nu.$$

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Lower Bound Property

$p^\star \geq -(1/4)\nu^\intercal A A^\intercal \nu - b^\intercal \nu \quad \forall \nu$

Deriving Waterfilling Expression

1. Form Lagrangian by relaxing $P_j > 0$:

$$\min_{P_j} \sum_j -B \log\left(1 + \frac{P_j}{\bar{P}}\gamma_j\right) p(\gamma_j) \to f_0(P_j)$$

$$\frac{\sum_j P_j}{\bar{P}} \leq 1 \Rightarrow \sum_j P_j - \bar{P} = 0 \to f_j(P_j)$$

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Hint: Write Lagrangian in terms of $P_j, \bar{P}, \gamma$ before taking derivative.

Extensions of waterfilling idea

Can be applied to any system where the sum of logarithms need to be optimized with a sum power and positivity constraints

Examples

- Continuous fading states
- Time-invariant frequency selective fading channel - waterfilling over frequency
- Time-varying frequency selective fading channel - waterfilling over time and frequency (may not be optimal)
- MIMO channels - waterfilling over spatial diversity

Block fading vs. frequency-selective fading

Block Fading:

$$\sum_{\gamma_i \geq \gamma_0} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) p(\gamma_i) = 1$$

Frequency-selective Fading:

$$\sum_{\gamma_i \geq \gamma_0} \left(\frac{1}{\gamma_0} - \frac{1}{\gamma_i} \right) = 1$$

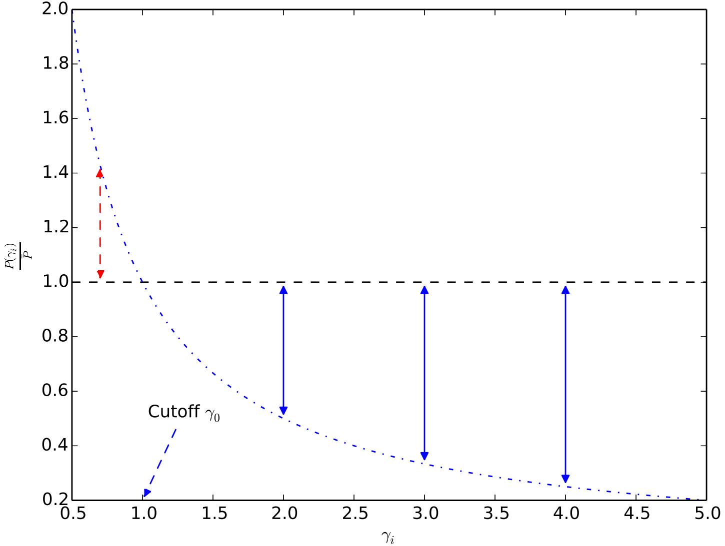
Finding the optimal power allocation

1. Assume $P(\gamma_i) > 0 \quad \forall i$
2. Solve for γ_0:

$$\frac{1}{\gamma_0} = 1 + \sum_i \frac{p(\gamma_i)}{\gamma_i} \qquad \text{or} \qquad \frac{N}{\gamma_0} = 1 + \sum_i \frac{1}{\gamma_i}$$

3. If any $P(\gamma_i) < 0$ assume the $P(\gamma_i)$ for lowest γ_i is zero and repeat previous step
4. Given γ_0, can compute $P(\gamma_i)$ and C.

"Waterfilling" interpretation of the solution $P(\gamma_i)$



Suboptimal power adaptation schemes

Power adaptation buys you little in practice.

Other Ideas

- Fix TX power but vary rate
- Channel inversion: Received SNR is constant
- Truncated channel inversion: Received SNR is constant, and do not use channel if gain is too low

Channel inversion in more details

Channel inversion

- If target SNR is σ, transmit at $\frac{\sigma}{\gamma}$
- Expected power constraint gives $\sigma = \frac{\bar{P}}{E[1/\gamma]}$
- What happens for Rayleigh fading?

Channel inversion in more details

Channel inversion

- If target SNR is σ, transmit at $\frac{\sigma}{\gamma}$
- Expected power constraint gives $\sigma = \frac{\bar{P}}{E[1/\gamma]}$
- What happens for Rayleigh fading? $E[1/\gamma] = \infty$

Truncated channel inversion

- Do not use the channel if $\gamma < \gamma_1$ (outage)
- If target SNR is σ, transmit at $\frac{\sigma}{\gamma}$
- Expected power constraint gives $\sigma = \frac{\bar{P}}{E_{\gamma > \gamma_1}[1/\gamma]}$

Question

How does capacity under truncated inversion behave with increasing outage probability ?