EE359 Discussion Session 5
Performance of Linear Modulation in Fading, Diversity

November 2, 2016
Outline

1. Recap
2. Performance analysis of linear modulation
3. Linear modulation in fading
4. Diversity
Announcement

- Midterm review on Tuesday, November 8, 4-6 pm, in Packard 364
- Midterm on Thursday, November 10, 6-8 pm, in 200-303
- OH hour changed for next week
Last discussion session

- Capacity formulas with CSIT and CSIR
- Optimal power and rate adaptation policies
- Suboptimal power adaptation policies

This session

- Performance analysis of linear modulation in fading
- Diversity
Outline

1 Recap

2 Performance analysis of linear modulation

3 Linear modulation in fading

4 Diversity
What is linear modulation

Definition

Any modulation where the data is encoded in real or complex symbols (i.e. in amplitude or phase) is linear modulation.

Observations

- Performance dependent on constellation (*not* on baseband waveform)
- Examples are BPSK, QPSK, MPAM, MQAM, etc.
- FSK (frequency shift keying) is not linear, why?

From now on we consider $P_s$ or $P_b$ as metric.
Linear modulation in AWGN

**Idea**

Noise is additive, hence

\[ y[i] = x[i] + n[i] \]

- \( P_b = Q(\sqrt{2\gamma_b}) \) if \( n[i] \sim \mathcal{N}(0, 1) \), and \( x[i] \in \text{BPSK} \)
- In general, well approximated by \( P_s \approx \alpha_M Q(\sqrt{\beta_M \gamma}) \), where \( \alpha_M, \beta_M \) are constellation dependent
- For differential PSK (DPSK) systems, \( P_b = \frac{1}{2} e^{-\gamma_b} \)
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Performance in fading

System model

\[ y[i] = \sqrt{\gamma[i]} x[i] + n[i] \]

Different metrics

- Average probability of error: Relevant when channel is fast fading
- Outage probability: Relevant when channel is slow fading
- Combined Outage + Avg. probability of error: shadowing (slow) and fading (fast)
Homework 5

Problem 1
Outage probability relevant when symbol time is much less than coherence time of channel
Computing the average probability of error

Idea
- Integrate the $Q$ function over fading distributions
- Use change of integration order to try to get closed form expressions

Some useful relations
- $P_b$ for BPSK in Rayleigh $\approx \frac{1}{4\gamma}$ (Closed form also possible)
- $P_b$ for DPSK in Rayleigh $\approx \frac{1}{2\gamma}$ (Closed form possible)
- $Q(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \int_{0}^{\pi/2} e^{-x^2/(2\sin^2\phi)} d\phi$
- Average BER computation is often MGF computation as $Q(\sqrt{\gamma}) \leq \frac{1}{2} e^{-\gamma/2}$
\( \bar{P}_s \) using MGF \( (\mathcal{M}_\gamma(s) = \int_0^\infty e^{s\gamma} p(\gamma) d\gamma) \)

**Idea**

Use fact that

\[
Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \frac{1}{\pi} \int_0^{\pi/2} e^{-x^2/2} \sin^2 \phi \, d\phi \leq \frac{1}{2} e^{-x^2/2}
\]

\[
\bar{P}_s \approx \int_0^\infty \frac{\alpha M}{\pi} Q(\sqrt{\beta M \gamma}) p(\gamma) d\gamma
\]

\[
= \frac{\alpha M}{\pi} \int_0^\infty \int_0^{\pi/2} e^{-\beta M \gamma/2} \sin^2 \phi \, p(\gamma) \, d\phi \, d\gamma
\]

\[
= \frac{\alpha M}{\pi} \int_0^{\phi=\pi/2} \int_0^{\gamma=\infty} e^{-\beta M \gamma/2} \sin^2 \phi \, p(\gamma) \, d\gamma \, d\phi
\]

\[
= \frac{\alpha M}{\pi} \int_{\phi=0}^{\phi=\pi/2} \mathcal{M}_\gamma(-\beta M/2 \sin^2 \phi) \, d\phi
\]
Combined outage and average error probability

Setting
Shadowing time scales are small

Idea
Three SNRs:
- $\gamma_s$: Instantaneous (random)
- $\bar{\gamma}_s$: Averaged over multipath fading (random)
- $\bar{\bar{\gamma}}_s$: Averaged over multipath fading and shadowing (influenced by e.g. path loss)
Outage in multipath fading

Setting

Shadowing time scales are large

Idea

- $\bar{P}_s$: shadowing time scales large for $\bar{P}_s$ to be relevant
- $\gamma_s$: Instantaneous (random)
- $\bar{\gamma}_s$: Averaged over multipath fading (usually fixed)
Question, for DPSK

Formulas

- \( P_b = \frac{1}{2} e^{-\gamma_b} \)
- \( \bar{P}_b = \frac{1}{2(1+\gamma_b)} \) in Rayleigh fading
- \( P_{\text{out}} = 1 - e^{-\gamma_0/\bar{\gamma}_b} \) in Rayleigh fading at level \( \gamma_0 \)
- \( P_{\text{out}} = Q \left( \frac{P_r - P_{\text{target}}}{\sigma} \right) \) due to shadowing with variance \( \sigma^2 \)

Question

What is the SNR for a given \( P_{\text{out}} \)?

Answer: It depends!

- In combined outage and average: Outage is due to shadowing, use fourth formula
- In fading: Outage is due to fading, use third formula
Question, for DPSK

Formulas

- \( P_b = \frac{1}{2} e^{-\gamma_b} \)
- \( \bar{P}_b = \frac{1}{2(1+\gamma_b)} \) in Rayleigh fading
- \( P_{out} = 1 - e^{-\gamma_0/\bar{\gamma}_b} \) in Rayleigh fading at level \( \gamma_0 \)
- \( P_{out} = Q \left( \frac{P_r - P_{target}}{\sigma} \right) \) due to shadowing with variance \( \sigma^2 \)

Question

What is the \( \gamma_b \) (or \( \bar{\gamma}_b \)) for a given \( P_s \) (or \( \bar{P}_s \))?  

Answer: It depends!

- In combined outage and average: \( \bar{P}_s \) is averaged over fading, use second formula
- In multipath fading: \( P_s \) is due to instantaneous SNR, use first formula
On error floors

Idea
As $\gamma_s \to \infty$, $P_{\text{error}} \to 0$ usually. Not true if there is an error floor!

Some reasons
- Differential modulation with large symbol times and/or fast fading (due to small $T_c$)
- Due to intersymbol interference ISI (or small $B_c$) $P_b \approx \left( \frac{\sigma}{T_s} \right)^2$

Some factors
- Correlation function of channel (channel coherence time $T_c$ and bandwidth $B_c$)
- Fading statistics, symbol time $T_s$

Question
What happens to error floors if $T_s$ decreases or data rate increases?
Problem 2

- Use average error probability requirement to get $\bar{\gamma}_b$
- Use $P_{\text{out}}$ requirement to set the target power in the presence of shadowing and path loss
- Use cell coverage area formula, if appropriate

Problem 3

Use formula for Rayleigh fading with DPSK, i.e. (6.90), with $K = 0$, with the $\rho_C$ function given by the Jakes’ formula (uniform scattering). This gives error floor due to doppler.
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Diversity

Idea

Use of independent fading realizations can reduce the probability of error/outage events

Some observations

- Diversity can be in time, space, frequency, polarization, ...
- Diversity order used as a measure of diversity, defined as
  \[ M = \lim_{\bar{\gamma} \to \infty} \frac{-\log P_e}{\log \bar{\gamma}} , \quad P_e = \bar{P}_s \text{ or } P_{out} \]
- Can also use array gain (or SNR gain) \( \bar{\gamma}_\Sigma / \bar{\gamma} \), where \( \bar{\gamma}_\Sigma \) is the average SNR after “diversity combining”
Diversity order

Specifying diversity order $M$ is roughly equivalent to saying that at high $\bar{\gamma}$,

$$\bar{P}_e \approx (\bar{\gamma})^{-M}$$

Array gain

Array gain $A_g$ is equivalent to ratio of average SNRs after diversity combining

$$A_g = \frac{\bar{\gamma}\Sigma}{\bar{\gamma}}$$
Diversity combining techniques

In this class, we have talked about two schemes to exploit diversity, both at the receiver.

**System model**

\[ r = \gamma x + n \]

**Some receiver diversity combining schemes**

- **Selection combining**: Choose the largest SNR of the independent realizations
- **Maximal ratio combining**: Combine all the independent received SNRs to maximize SNR
Selection combining (SC)

**Idea**

Given $M$ i.i.d. r.v., $\gamma_1, \ldots, \gamma_M \geq 0$,

$$P(\max_i (\gamma_i) < c) = P(\gamma_i < c)^M$$

**Some observations**

- Define $\gamma_\Sigma = \max_i \gamma_i$
- In Rayleigh fading $\bar{\gamma}_\Sigma = \bar{\gamma}(\sum_{i=1}^{M} 1/i)$ ($\bar{\gamma}$: average SNR at a branch)
- $P_b$ in general difficult, but for DPSK and Rayleigh fading,

$$P_b = M/2 \sum_{m=0}^{M-1} (-1)^m \frac{(M-1)}{1 + m + \bar{\gamma}}$$
Outage probability

\[ P_{\text{out}} = \left( 1 - e^{-\frac{\gamma_0}{\bar{\gamma}}} \right)^M \]

Question (SC in Rayleigh fading)

- What is the diversity gain?: \( M \)
- What is the SNR gain?: \( \sum_{i}^{M} \frac{1}{i} \)
Maximal ratio combining (MRC)

Idea

Instead of discarding weaker branches, combine the SNRs of all branches, i.e.

\[ \gamma_\Sigma = \sum_{i=1}^{M} \gamma_i \]

Nuts and bolts

- Need to make the received components of the same phase (not a problem with modern DSP)
- Maximal ratio combining maximizes received SNR, i.e. solves the following problem

\[
\max_{a: \|a\|^2 = 1} \frac{E[|a^H \gamma x|^2]}{E[|a^H n|^2]}
\]

- MGF of sums decompose into product of individual MGFs so easy to analyse \( \bar{P}_s \)
MRC continued (Outage probability and $\bar{P}_s$)

Outage probability

$$P_{\text{out}} = 1 - e^{\frac{\gamma_0}{\bar{\gamma}}} \left( \sum_{i=0}^{M-1} \left( \frac{\gamma_0}{\bar{\gamma}} \right)^i / i! \right)$$

Average probability of error $\bar{P}_s$

- The MGF of sum decouples into product of MGFs
- For DPSK and Rayleigh fading, average error probability is

$$\frac{1}{2} E_{\gamma_{\Sigma}}[e^{-\gamma_{\Sigma}}] = \frac{1}{2} \prod_{i=1}^{M} E_{\gamma_i}[e^{-\gamma_i}] = \frac{1}{2} \prod_{i=1}^{M} \mathcal{M}(-1)$$

- For general constellations $\bar{P}_s$ is approximately of the form

$$C \int_{\phi=A}^{\phi=B} (\mathcal{M}(-\gamma/2 \sin^2 \phi))^M d\phi$$
Questions

- What is the diversity order for MRC?: $M$
- What is the SNR gain for MRC?: $M$
Problem 4
Concepts in SC (Selection combining) - probability of outage

Problem 5
Concepts in MRC - Deriving optimal weights, computing $\hat{P}_s$. Using MGF approach in part (c).

Problem 6
Notions of combining - SC/MRC. For part (c), if $A \Rightarrow B$ or in other words $A \subseteq B$, then $P(A) \leq P(B)$.  

Problem 7

Diversity gain in SC and MRC at high/low SNR.