1. Problem 1 (50 points): Short Answer

(a) (10 pts) Consider a 2x2 MIMO system with SNR per spatial dimension of $\gamma_1 = \sigma_1^2 \rho$ and $\gamma_2 = \sigma_2^2 \rho$ for MIMO matrix singular values $\sigma_i$, $i = 1, 2$ and $\rho = P/\sigma_n^2$ where $P$ is total transmit power and $\sigma_n^2$ is the noise power. Find the relationship between $\gamma_1$ and $\gamma_2$ such that beamforming is capacity-achieving for this system.

(b) (15 pts) In OFDM systems we have seen that when the length of the cyclic prefix $\mu$ equals the length of the multipath delay spread $T_m$, there is no ISI between OFDM blocks. But the length of the multipath delay spread $T_m$ is random, while the length $\mu$ is fixed as part of the OFDM system design. Suppose for a given multipath channel, $\mu = T_m / 2$, i.e. the cyclic prefix is too short: will the OFDM blocks experience ISI and, if so, how many OFDM time symbols will be affected? Suppose now that for a given multipath channel, $\mu = 2T_m$, how is the system performance different than if the cyclic prefix was designed such that $\mu = T_m$?

(c) (10 pts) Consider a short spreading code for a binary spread spectrum system that repeats every bit time $T_b = 100T_c$ for $N = 100$ chips/bit. What is the ISI power reduction for a maximal linear code in the two-path channel model with the despreadi ng perfectly synchronized to the LOS component and the delay of the 2nd path equal to $T_b/2$? What if the delay is $0.999T_b$?

(d) (15 pts) Give two possible reasons why next-generation WiFi and cellular systems have chosen OFDM over spread spectrum for their design. What is one advantage of spread spectrum over OFDM? For what constraints on a given system would an equalizer be used to compensate for ISI rather than either spread spectrum or OFDM?

2. Problem 2 (60 points): Miley Cyrus and MIMO Communications

Miley is about to release her latest music video “Breaking Wall” and wants to make sure it gets downloaded with the highest quality. In between filming segments, she spends her spare time reading about MIMO communication, and knows that optimally using different spatial dimensions are her best bet to get the best quality downloads. In particular, she knows that MIMO precoding is a technique that sits between full spatial multiplexing (using all spatial dimensions of the MIMO channel) and beamforming (using only the strongest spatial dimension). In this problem we investigate how Miley adapts both the use of different spatial dimensions and the modulation and power on each dimension to ensure each fan get every detail of her “Breaking Wall” video. Suppose her #1 fan has a $3 \times 3$ MIMO system for the video download with the following channel matrix

$$
H = \begin{bmatrix}
0.1 & 0.3 & 0.7 \\
0.5 & 0.4 & 0.1 \\
0.2 & 0.6 & 0.8
\end{bmatrix}
$$
This can be written via the SVD as
\[
H = \begin{bmatrix}
-0.555 & 0.3764 & -0.7418 \\
-0.338 & -0.9176 & -0.2158 \\
-0.7619 & 0.1278 & 0.6349
\end{bmatrix}
\begin{bmatrix}
1.3333 & 0 & 0 \\
0 & 0.5129 & 0 \\
0 & 0 & 0.0965
\end{bmatrix}
\begin{bmatrix}
-0.2811 & -0.7713 & -0.5710 \\
-0.5679 & -0.3459 & 0.7469 \\
-0.7736 & 0.5342 & -0.3408
\end{bmatrix}
\]

In beamforming Miley transmits along the spatial dimension with the largest singular value. In spatial multiplexing she transmits along all three spatial dimensions. 2D precoding allows her to transmit along two of the three spatial dimensions, corresponding to the two largest singular values. This is done using transmit precoding and receiver shaping matrices corresponding to the first 2 columns of \( V \) and \( U \) in the SVD of \( H \). Assume the system bandwidth is \( B = 1 \) MHz and noise power is 0 dBm. You can use the BER approximation in AWGN of \( BER \approx 2e^{-1.5\gamma/(M-1)} \) for your calculations.

(a) (6 pts) What are the transmit precoding and receiver shaping matrices associated with beamforming (1 spatial stream) and 2D precoding (2 spatial streams)?

(b) (15 pts) Find the capacity of this channel for a total transmit power of 10 dBm (when Miley is power-hungry) and for a total transmit power of 0dBm (when she is going green). What number of streams should Miley use to achieve capacity in each case? Explain why changing the transmit power changes the optimal number of streams that achieve capacity.

(c) (12 pts) Find the data rate that can be achieved with optimal adaptive modulation across spatial dimensions for a total transmit power of 20 dBm, assuming unconstrained MQAM modulation and a target BER of \( 10^{-4} \).

(d) (15 pts) For a transmit power of 20 dBm, and MQAM constellations of no transmission, BPSK, or \( M = 2^k, k = 2, 3, 4, ..., \) a target BER not exceeding \( 10^{-4} \) and power equally divided among all spatial streams, find the total data rate associated with all data streams under beamforming (1 spatial stream), 2D precoding (2 spatial streams), and spatial multiplexing (3 spatial streams). What is the best way for Miley to use her spatial dimensions to maximize rate? How much rate penalty does she pay given the fixed-power-per-dimension and the square MQAM modulation constraints relative to the unconstrained system considered in part (c)?

(e) (12 pts) For 16QAM modulation and a 20dBm transmit power equally divided among all spatial streams, find the probability of error for each stream under beamforming (1 spatial stream), precoding (2 spatial streams), and spatial multiplexing (3 spatial streams), and the average probability of error for all transmitted data in each case.

3. Problem 3 (30 points): Adaptive OFDM: Maximizing capacity for fun and profit!

You are building a startup focused on OFDM technology. Your OFDM system has \( N \) subchannels with constant fading on each subchannel. You use a cyclic prefix is used to remove ISI between FFT blocks such that the \( i \)th subchannel can be represented by \( Y[i] = X[i]H[i] + N[i] \), where \( H[i] \) is the fading associated with this subchannel.

(a) (7 pts) Suppose the delay spread on the channel is \( 10 \mu s \). If the total channel bandwidth is 10 MHz, and the OFDM system has an FFT that is a power of 2, what size FFT in the OFDM implementation ensures approximately flat fading on each subchannel (i.e. a subchannel bandwidth less than the channel coherence bandwidth)?

(b) (10 pts) Assume an OFDM system with 8 subchannels each of bandwidth 100 KHz. The total transmit power for the system is \( P = 400 \) mW and with 400 mW transmitted on each subchannel, the received SNR on each subchannel is \( \gamma_i = 400/i \) (linear units), \( i = 1, ..., 8 \). Due to complexity and power amplifier constraints, your startup’s system divides the transmit power of 400 mW equally divided among all the subchannels. What is the capacity of your OFDM system?
(c) (13 pts) Your competition adapts the transmit power so that the SNR per subchannel is constant. What is the capacity under this adaptive power policy?

4. Problem 4 (60 points): Spreading Hillary Clinton Poll Numbers

Hillary Clinton is running for president and wants to check the latest polls about her chances. This problem illustrates how Hillary’s use of a spread spectrum phone with a RAKE receiver helps to get her poll numbers with higher reliability. Our hope-to-be president is in her New York mansion checking the polls on her cell phone, which has a multipath channel to her 3G (wideband CDMA) base station with impulse response

\[ h(t) = \alpha_0 \delta(t - \tau_0) + \alpha_1 \delta(t - \tau_1). \]

The \( \alpha_i \) are Rayleigh fading coefficients with \( E[\alpha_0^2] = 100 \) and \( E[\alpha_1^2] = 40 \). The transmit power and noise power are such that a spread spectrum receiver locked to the \( i \)th multipath component will have an SNR of \( \alpha_i^2 \) in the absence of the other multipath components. The number of chips per bit is 100 and the code has autocorrelation

\[ \rho_c(\tau) = \begin{cases} 1 - |\tau|/T_c & |\tau| \leq T_c \\ 0 & T_c \leq |\tau| \leq (N - 1)T_c \end{cases} \]

which is periodic every \( T_b = 100T_c \). Assume \( \tau_0 = 0 \) and \( \tau_1 = 5T_c \).

(a) (10 pts) Suppose only the LOS component \( \alpha_0 \) is nonzero. Neglecting the effects of noise, if the synchronization loop starts out offset by 15.5\( T_c \) from the delay \( \tau_0 = 0 \) of the LOS path, and if the synchronization loop adjusts its timing offset monotonically in increments of .5\( T_c \), how many steps are needed to lock onto the correct timing? What is the maximum possible synchronization time for the synchronization loop to lock onto the autocorrelation peak? If the synchronization loop locks to the autocorrelation peak at a delay of 100\( T_c \), how would this affect performance?

(b) (10 pts) Hillary is using a standard non-RAKE receiver to check her poll numbers, which are transmitted using DPSK modulation. What is the outage probability of her signal at an instantaneous \( P_b = 10^{-4} \), assuming she is locked to the 1st multipath component (with delay \( \tau_1 = 5T_c \)) at a timing offset of .1\( T_c \)?

(c) (15 pts) Hillary’s smart assistant took EE359 and hands her a phone with a 2-branch RAKE receiver that uses selection combining of the 2 branches. Find the outage probability of Hillary’s DPSK signal with this new receiver at an instantaneous \( P_b = 10^{-4} \) assuming a timing offset of .1\( T_c \) in each branch.

(d) (15 pts) Hillary’s smarter assistant took EE359 and got an A+. She hands Hillary a phone with a 2-branch RAKE receiver that uses maximal-ratio combining. Find the outage probability of Hillary’s signal with this receiver at an instantaneous \( P_b = 10^{-4} \) assuming a timing offset of .1\( T_c \) in each branch.

(e) (10 pts) For the 2-branch MRC RAKE receiver and .1\( T_c \) timing offset of part (d), find the average probability of error for Hillary’s signal with this phone.