EE359, Wireless Communications, Fall 2017
Homework 6 (100 pts)

Due: Friday (November 17), 4 pm

Please refer to the homework page on the website (ee359.stanford.edu/homework) for guidelines.

1. (20 pts) **Optimal variable rate variable power MQAM** (Problem 9-9): Consider a discrete time-varying AWGN channel with four channel states. Assuming a fixed transmit power $S$, the received SNR associated with each channel state is $\gamma_1 = 5$ dB, $\gamma_2 = 10$ dB, $\gamma_3 = 15$ dB, and $\gamma_4 = 20$ dB, respectively. The probabilities associated with the channel states are $p(\gamma_1) = 0.4$ and $p(\gamma_2) = p(\gamma_3) = p(\gamma_4) = 0.2$. Also, the target $P_b = 10^{-3}$.

   (a) (10 pts) Find the optimal power and rate adaptation for continuous-rate adaptive MQAM on this channel.

   (b) (5 pts) Find the average spectral efficiency with this optimal adaptation.

   (c) (5 pts) Find the truncated channel inversion power control policy for this channel and the maximum data rate that can be supported with this policy.

2. (25 pts) **Adaptive MQAM with discrete constellations** (Problem 9-10, 9-11): Consider a Rayleigh fading channel with an average received SNR of 20 dB, a Doppler frequency of 80 Hz, and a required BER of $10^{-3}$.

   (a) (10 pts) Find the spectral efficiency of this channel using truncated channel inversion, assuming the constellation is restricted to size 0, 2, 4, 16, 64, or 256. Please attach any programming script you write to solve this.

   (b) (10 pts) Suppose you use adaptive MQAM modulation on this channel with constellations restricted to size 0, 2, 4, 16, and 64. Using $\gamma^*_K = 0.1$, find the fading regions $R_j$ associated with each of these constellations. Also find the average spectral efficiency of this restricted adaptive modulation scheme.

   (c) (5 pts) Does the data rate increase as $\gamma^*_K$ increases? Does the transmit power associated with a given $\gamma$ to meet the BER target increase or decrease as $\gamma^*_K$ increases?

3. (15 pts) **Estimation error in adaptive modulation schemes** (Problem 9-12*): Consider a Rayleigh fading channel with an average received SNR of 20 dB, a signal bandwidth of 30 kHz, a Doppler frequency of 80 Hz, and a required BER of $10^{-3}$. For this problem, please review Section 9.3.7 (Channel Estimation Error and Delay) of the text.

   (a) Assume the SNR estimate at the transmitter $\hat{\gamma}$ has the same distribution as the true channel SNR $\gamma$, so $p(\hat{\gamma}) = p(\gamma)$. For the optimal variable-rate variable-power MQAM scheme with no restrictions on rate or power, suppose that the transmit power and rate is adapted relative to $\hat{\gamma}$ instead of $\gamma$. Will the average transmitted data rate and average transmit power be larger, smaller, or the same as under perfect channel estimates ($\hat{\gamma} = \gamma$), and why? If over a given symbol time $\hat{\gamma} > \gamma$, will the probability of error associated with that symbol be larger or smaller than the target value and why?
(b) Suppose the estimation error \( \epsilon = \frac{\hat{\gamma}}{\gamma} \) in a variable-rate variable-power MQAM system with a target BER of \( 10^{-3} \) is uniformly distributed between .5 and 1.5. Find the resulting average probability of bit error for this system.

(c) Find an expression for the average probability of error in a variable-rate variable-power MQAM system in which the SNR estimate \( \hat{\gamma} \) available at the transmitter is both a delayed and noisy estimate of \( \gamma : \hat{\gamma}(t) = \gamma(t-\tau) + \gamma\epsilon(t) \). What joint distribution is needed to compute this average?

4. (15 pts) Calculating errors due to estimation and delay for adaptive modulation: In this problem, we investigate the effect of channel estimation errors and delays in system performance. Consider a Rayleigh fading channel with mean power 1, uniform scattering and doppler of \( f_D \) Hz. The channel output \( y[t] \) is thus given by

\[
y[t] = h[t]x[t] + \nu[t],
\]

where \( h[t] \) is the fading realization, \( x[t] \) is the transmitted symbol and \( \nu[t] \) is the additive noise with power 1. Symbol time \( T_b \) s. We use adaptive modulation with no transmission, BPSK or 4-QAM; and the receiver feeds back the receiver output to the transmitter at the end of every symbol time. The transmitter uses \( y[t] \) and \( x[t] \) to estimate \( \hat{h}[t+1] \) and uses this estimate to choose the constellation scheme to either have an outage or maintain a constant BER \( P_b \) for time \( t+1 \). The receiver knows the instantaneous channel state information \( h[t] \) and uses that for decoding.

(a) (3 pts) Find the two threshold received SNRs \( \gamma_0, \gamma_1 (\gamma_0 < \gamma_1) \) at which the transmitter switches modulations. Use the exact AWGN error probability expressions wherever possible.

(b) (3 pts) Find out the MMSE (minimum mean squared error) estimate of \( h[t] \) given \( y[t] \) and \( x[t] \). This is the effect of the channel estimation error.

(c) (3 pts) Find out the MMSE estimate of \( h[t+1] \) given the estimate of \( h[t] \). This is the effect of delay in feedback.

(d) (6 pts) Assuming that the transmitter uses the MMSE estimate \( \hat{h}[t+1] \) to choose the transmit modulation for time \( t+1 \), and the actual realization is \( h[t+1] \) known to the receiver, write down an expression for the bit error rate \( P_b(\|h[t+1]\|,\|\hat{h}[t+1]\|) \). Further, write the expression for the BER estimated over the joint distribution of \( y[t] \) and \( h[t+1] \).

Note: In general the optimal transmitter would want to use past values of the feedback to refine its estimate of \( h[t+1] \) but in this problem we do not consider that possibility.

5. (15 pts) Equivalent MIMO capacity expressions (Problems 10-5 & 10-7):

(a) (5 pts) The capacity of a static MIMO channel with only receiver CSI is given by \( C = \sum_{i=1}^{R_H} B \log_2(1 + \sigma_i^2 \rho/M_t) \). Show that, if the sum of squares of singular values \( (\sigma_i) \) is bounded, then this expression is maximized when all \( R_H \) singular values are equal.

(b) (10 pts) Use properties of the SVD to show that, for a MIMO channel that is known to the transmitter and receiver both, the general capacity expression

\[
C = \max_{R_x : \text{Tr}(R_x) = \rho} B \log_2 \det[I_{M_t} + HR_x H^H]
\]

reduces to

\[
C = \max_{\rho : \sum_i \rho_i \leq \rho} \sum_i B \log_2(1 + \sigma_i^2 \rho_i)
\]

for singular values \( \{\sigma_i\} \) and SNR \( \rho \).
6. (10 pts) **Capacity of Massive-MIMO systems** (Problem 10-9 modified): Assume a ZMCSCG MIMO system with channel matrix \( \mathbf{H} \in \mathbb{R}^{M_t \times M_r} \) corresponding to \( M_t \) transmit and \( M_r \) receive antennas (\( \mathbb{E}[|H_{i,j}|^2] = 1 \) for all \( i \) and \( j \)). Show using the law of large numbers that

\[
\lim_{M_r \to \infty} \frac{1}{M_r} \mathbf{H}^H \mathbf{H} = \mathbf{I}_{M_t}.
\]

Then use this to show that

\[
\lim_{M_r \to \infty} B \log_2 \det \left( \mathbf{I}_{M_t} + \frac{\rho}{M_r} \mathbf{H}^H \mathbf{H} \right) = M_t B \log_2(1 + \rho).
\]