Announcements
- Makeup lecture Feb 2, 5-6:15.
- Presentation schedule will be sent out later today, presentations will start 1/30.
- Next lecture: Random/Multiple Access, SS, MUD

Capacity of Broadcast ISI Channels

Capacity of MAC Channels
- In AWGN
- In Fading and ISI

Duality between the MAC and the BC

Capacity of MIMO Multiuser Channels
Review of Last Lecture

- Channel capacity region of broadcast channels
- Capacity in AWGN
  - Use superposition coding and optimal power allocation
- Capacity in fading
  - Ergodic capacity: optimally allocate resources over time
  - Outage capacity: maintain fixed rates in all states
  - Minimum rate capacity: fixed min. rate in all states, use excess resources to optimize average rate above min.
Broadcast Channels with ISI

- ISI introduces memory into the channel

- The optimal coding strategy decomposes the channel into parallel broadcast channels
  - Superposition coding is applied to each subchannel.

- Power must be optimized across subchannels and between users in each subchannel.
Both $H_1$ and $H_2$ are finite IR filters of length $m$.

The $w_{1k}$ and $w_{2k}$ are correlated noise samples.

For $1 < k \leq n$, we call this channel the n-block discrete Gaussian broadcast channel (n-DGBC).

The channel capacity region is $C=(R_1,R_2)$. 
Equivalent Parallel Channel Model
Channel Decomposition

• Via a DFT, the BC with ISI approximately decomposes into \( n \) parallel AWGN degraded broadcast channels.
  - As \( n \) goes to infinity, this parallel model becomes exact

• The capacity region of parallel degraded broadcast channels was obtained by El-Gamal (1980)
  - Optimal power allocation obtained by Hughes-Hartogs (’75).

\[ \sum_{i=0}^{n-1} E[x_i^2] \leq nP \] on the original channel is converted by Parseval’s theorem to \( \sum_{i=0}^{n-1} E[(X_i')^2] \leq n^2 P \) on the equivalent channel.
Capacity Region of Parallel Set

- Achievable Rates (no common information)

\[
\left\{ R_1 \leq 0.5 \sum_{j: \sigma_1 j < \sigma_2 j} \log \left( 1 + \frac{\alpha_j P_j}{\sigma_1 j} \right) + 0.5 \sum_{j: \sigma_1 j \geq \sigma_2 j} \log \left( 1 + \frac{\alpha_j P_j}{1 - \alpha_j P_j + \sigma_1 j} \right), \right.

R_2 \leq 0.5 \sum_{j: \sigma_1 j < \sigma_2 j} \log \left( 1 + \frac{(1 - \alpha_j) P_j}{\alpha_j P_j + \sigma_2 j} \right) + 0.5 \sum_{j: \sigma_1 j \geq \sigma_2 j} \log \left( 1 + \frac{(1 - \alpha_j) P_j}{\sigma_2 j} \right),
\]

\[0 \leq \alpha_j \leq 1, \sum P_j \leq n^2 P \}

- Capacity Region

- For \(0 < \beta \leq \infty\) find \(\{\alpha_j\}, \{P\}\) to maximize \(R_1 + \beta R_2 + \lambda \sum P_j\).

- Let \((R_1^*, R_2^*)_{n, \beta}\) denote the corresponding rate pair.

- \(C_n = \{(R_1^*, R_2^*)_{n, \beta} : 0 < \beta \leq \infty\}, C = \liminf_{n \to \infty} C_n\).
Limiting Capacity Region

\( \{ R_1 \leq 0.5 \int_{f: H_1(f) > H_2(f)} \log \left( 1 + \frac{\alpha(f)P(f)|H_1(f)|^2}{0.5N_0} \right) + 0.5 \int_{f: H_1(f) \leq H_2(f)} \log \left( 1 + \frac{\alpha_jP_j}{(1-\alpha_j)P_j + \sigma_{1j}} \right) \}, \)

\( R_2 \leq 0.5 \int_{f: H_1(f) > H_2(f)} \log \left( 1 + \frac{(1-\alpha(f))P(f)}{\alpha(f)P(f) + 0.5N_0/H_2(f)^2} \right) + 0.5 \int_{f: H_1(f) \leq H_2(f)} \log \left( 1 + \frac{(1-\alpha(f))P(f)|H_2(f)|^2}{0.5N_0} \right), \)

\( 0 \leq \alpha(f) \leq 1, \quad \int P(f)df \leq P \} \)
Optimal Power Allocation: Two Level Water Filling

![Graphs showing two level water filling with different lambda values.]

- For \( \lambda_1 = \frac{2}{5}, \lambda_2 = \frac{3}{5} \):
  - Region P1(w) indicates power allocation.
  - Region P2(w) indicates power allocation.

- For \( \lambda_1 = \frac{1}{2}, \lambda_2 = \frac{1}{2} \):
  - Similar regions P1(w) and P2(w) are visible.

- For \( \lambda_1 = \frac{3}{5}, \lambda_2 = \frac{2}{5} \):
  - Regions P1(w) and P2(w) are shown with different water levels.
Capacity vs. Frequency
Capacity Region
Multiple Access Channel

- Multiple transmitters
  - Transmitter $i$ sends signal $X_i$ with power $P_i$
- Common receiver with AWGN of power $N_0B$
- Received signal:

$$Y = \sum_{i=1}^{M} X_i + N$$
MAC Capacity Region

- Closed convex hull of all \((R_1, \ldots, R_M)\) s.t.

\[
\sum_{i \in S} R_i \leq B \log \left[ 1 + \sum_{i \in S} \frac{P_i}{N_0B} \right], \quad \forall S \subseteq \{1, \ldots, M\}
\]

- For all subsets of users, rate sum equals that of 1 superuser with sum of powers from all users

- **Power Allocation and Decoding Order**
  - Each user has its own power (no power alloc.)
  - Decoding order depends on desired rate point
Two-User Region

\[ C_i = B \log\left[1 + \frac{P_i}{N_0 B}\right], \quad i = 1, 2 \]

\[ \hat{C}_1 = B \log\left[1 + \frac{P_1}{N_0 B + P_2}\right], \quad \hat{C}_2 = B \log\left[1 + \frac{P_2}{N_0 B + P_1}\right] \]
Fading and ISI

- MAC capacity under fading and ISI determined using similar techniques as for the BC

- In fading, can define ergodic, outage, and minimum rate capacity similar as in BC case
  - Ergodic capacity obtained based on AWGN MAC given fixed fading, averaged over fading statistics
  - Outage can be declared as common, or per user

- MAC capacity with ISI obtained by converting to equivalent parallel MAC channels over frequency
Characteristics

- Corner points achieved by 1 user operating at his maximum rate
  - Other users operate at rate which can be decoded perfectly and subtracted out (IC)
- Time sharing connects corner points
  - Can also achieve this line via rate splitting, where one user “splits” into virtual users
- FD has rate $R_i \leq B_i \log[1 + P_i/(N_0B)]$
- TD is straight line connecting end points
  - With variable power, it is the same as FD
- CD without IC is box
Fading MAC Channels

- Noise is AWGN with variance $\sigma^2$.
- Joint fading state (known at TX and RX):
  \[ y(n) = \sum_{i=1}^{M} \sqrt{h_i(n)} x_i(n) + z(n) \]

\[ h = (h_1(n), \ldots, h_M(n)) \]
Capacity Region*

- Rate allocation $R(h) \in \mathbb{R}^M$

- Power allocation $P(h) \in \mathbb{R}^M$
  - Subject to power constraints: $E_h[P(h)] \leq P$

- Boundary points: $R^*$
  - $\exists \lambda, \mu \in \mathbb{R}^M$ s.t. $[R(h), P(h)]$ solves

\[
\max \mu R - \lambda P \quad \text{s.t.} \sum_{i \in S} R_i \leq 0.5 \log \left[ 1 + \sum_{i \in S} h_i P_i \sigma^2 \right], \forall S \subseteq \{1, \ldots, M\}
\]

with $E_h[R_i(h)] = R_i^*$

*Tse/Hanly, 1996*
Unique Decoding Order*

• For every boundary point $R^*$:
  - There is a unique decoding order that is the same for every fading state
  - Decoding order is reverse order of the priorities
    \[
    \mu_1 \geq \cdots \geq \mu_M \Rightarrow \text{Decoding order: } M, M-1, \ldots, 1
    \]

• Implications:
  - Given decoding order, only need to optimally allocate power across fading states
  - Without unique decoding order, utility functions used to get optimal rate and power allocation

*S. Vishwanath
Characteristics of Optimum Power Allocation

- A user’s power in a given state depends only on:
  - His channel \( h_{ik} \)
  - Channels of users decoded just before \( (h_{ik-1}) \) and just after \( (h_{ik+1}) \)
  - Power increases with \( h_{ik} \) and decreases with \( h_{ik-1} \) and \( h_{ik+1} \)
  - Power allocation is a modified waterfilling, modified to interference from active users just before and just after

- User decoded first waterfills to SIR for all active users
Transmission Regions

- The region where no users transmit is a hypercube
  - Each user has a unique cutoff below which he does not transmit
- For highest priority user, always transmits above some $h_1^*$
- The lowest priority user, even with a great channel, doesn’t transmit if some other user has a relatively good channel

\[
\begin{align*}
\mu_1 > \mu_2 \\
P_1 &= 0, P_2 > 0 \\
P_1 > 0, P_2 > 0 \\
P_1 > 0, P_2 = 0 \\
\end{align*}
\]
Two User Example

- Power allocation for $\mu_1 > \mu_2$

\[ P_1(h) = \begin{cases} 
0 & h_1 < \frac{\lambda_1}{\mu_1} \\
\frac{\mu_1 - 1}{\lambda_1 h_1} & h_1 > \frac{\lambda_1}{\mu_1}, P_2(h) = 0 \\
\frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2 (h_1 / h_2)} - \frac{1}{h_1} & h_1 > \frac{\lambda_1}{\mu_1}, P_2(h) \neq 0 
\end{cases} \]

\[ P_2(h) = \begin{cases} 
0 & h_2 > \frac{\lambda_2}{\mu_2} \\
\frac{\mu_2}{\lambda_2} - \frac{1 + h_1 P_1(h)}{h_2} & \frac{h_2}{1 + h_1 P_1(h)} > \frac{\lambda_2}{\mu_2} \\
\frac{h_2}{1 + h_1 P_1(h)} & \frac{h_2}{1 + h_1 P_1(h)} < \frac{\lambda_2}{\mu_2} 
\end{cases} \]
Ergodic Capacity Summary

- Rate region boundary achieved via optimal allocation of power and decoding order

- For any boundary point, decoding order is the same for all states
  - Only depends on user priorities

- Optimal power allocation obtained via Lagrangian optimization
  - Only depends on users decoded just before and after
  - Power allocation is a modified waterfilling
  - Transmission regions have cutoff and critical values
MAC Channel with ISI*

- Use DFT Decomposition
- Obtain parallel MAC channels
- Must determine each user’s power allocation across subchannels and decoding order
- Capacity region no longer a pentagon

*Cheng and Verdu, IT’93
Optimal Power Allocation

- Capacity region boundary: maximize $\mu_1 R_1 + \mu_2 R_2$
- Decoding order based on priorities and channels
- Power allocation is a two-level water filling
  - Total power of both users is scaled water level
  - In non-overlapping region, best user gets all power (FD)
  - With overlap, power allocation and decoding order based on $\lambda$s and user channels.

\[ \frac{b_1}{|H_1(f)|^2} + \mu_1 \]
\[ \frac{b_2}{|H_2(f)|^2} + \mu_2 \]
Comparison of MAC and BC

- **Differences:**
  - Shared vs. individual power constraints
  - Near-far effect in MAC

- **Similarities:**
  - Optimal BC “superposition” coding is also optimal for MAC (sum of Gaussian codewords)
  - Both decoders exploit successive decoding and interference cancellation
MAC-BC Capacity Regions

- MAC capacity region known for many cases
  - Convex optimization problem

- BC capacity region typically only known for (parallel) degraded channels
  - Formulas often not convex

- Can we find a connection between the BC and MAC capacity regions?

Duality
Dual Broadcast and MAC Channels

Gaussian BC and MAC with *same* channel gains and *same* noise power at each receiver

Broadcast Channel (BC)  Multiple-Access Channel (MAC)
The BC from the MAC

\[ C_{MAC}(P_1, P_2; h_1, h_2) \subseteq C_{BC}(P_1 + P_2; h_1, h_2) \]

\( h_1 > h_2 \)

Blue = BC

Red = MAC

Case 1: \( P_1=0.5, P_2=1.5 \)

Case 2: \( P_1=1, P_2=1 \)

Case 3: \( P_1=1.5, P_2=0.5 \)

\[ C_{BC}(P; h_1, h_2) = \bigcup_{0 \leq P_1 \leq P} C_{MAC}(P_1, P - P_1; h_1, h_2) \]
Sum-Power MAC

\[ C_{BC}(P; h_1, h_2) = \bigcup_{0 \leq P_1 \leq P} C_{MAC}(P_1, P - P_1; h_1, h_2) \equiv C_{MAC}^{Sum}(P; h_1, h_2) \]

- MAC with sum power constraint
  - Power pooled between MAC transmitters
  - No transmitter coordination

Same capacity region!
BC to MAC: Channel Scaling

- Scale channel gain by $\sqrt{\alpha}$, power by $1/\alpha$
- MAC capacity region unaffected by scaling
- Scaled MAC capacity region is a subset of the scaled BC capacity region for any $\alpha$
- MAC region inside scaled BC region for any scaling

\[
\begin{align*}
\frac{P_1}{\alpha} & + P_2 \\
\sqrt{\alpha h_1} & \\
\sqrt{\alpha h_1} & \\
h_2 & \quad h_2
\end{align*}
\]
The BC from the MAC

Blue = Scaled BC
Red = MAC

\[ C_{MAC} (P_1, P_2; h_1, h_2) = \bigcap_{\alpha > 0} C_{BC} \left( \frac{P_1}{\alpha} + P_2; \sqrt{\alpha h_1}, h_2 \right) \]
Duality: Constant AWGN Channels

- **BC in terms of MAC**

\[ C_{BC}(P; h_1, h_2) = \bigcup_{0 \leq P_1 \leq P} C_{MAC}(P_1, P - P_1; h_1, h_2) \]

- **MAC in terms of BC**

\[ C_{MAC}(P_1, P_2; h_1, h_2) = \bigcap_{\alpha > 0} C_{BC}\left(\frac{P_1}{\alpha} + P_2; \alpha h_1, h_2\right) \]

*What is the relationship between the optimal transmission strategies?*
Transmission Strategy
Transformations

- Equate rates, solve for powers

\[ R_1^M = \log(1 + \frac{h_1^2 P_1^M}{h_2 P_2^M + \sigma^2}) = \log(1 + \frac{h_1^2 P_1^B}{\sigma^2}) = R_1^B \]

\[ R_2^M = \log(1 + \frac{h_2^2 P_2^M}{\sigma^2}) = \log(1 + \frac{h_2^2 P_2^B}{h_2^2 P_1^B + \sigma^2}) = R_2^B \]

- Opposite decoding order
  - Stronger user (User 1) decoded last in BC
  - Weaker user (User 2) decoded last in MAC
Duality Applies to Different Fading Channel Capacities

- **Ergodic (Shannon) capacity**: maximum rate averaged over all fading states.

- **Zero-outage capacity**: maximum rate that can be maintained in all fading states.

- **Outage capacity**: maximum rate that can be maintained in all nonoutage fading states.

- **Minimum rate capacity**: Minimum rate maintained in all states, maximize average rate in excess of minimum

Explicit transformations between transmission strategies
Duality: Minimum Rate Capacity

- BC region known
- MAC region can only be obtained by duality

What other capacity regions can be obtained by duality?

Broadcast MIMO Channels
Broadcast MIMO Channel

\[ y_1 = H_1 x + n_1 \]
\[ y_2 = H_2 x + n_2 \]

\[ n_1 \sim N(0, I_{r_1}) \quad n_2 \sim N(0, I_{r_2}) \]

Non-degraded broadcast channel

\[ \begin{align*}
  t \geq 1 & \quad \text{TX antennas} \\
  r_1 \geq 1, \ r_2 \geq 1 & \quad \text{RX antennas}
\end{align*} \]
Dirty Paper Coding (Costa’83)

- Basic premise
  - If the interference is known, channel capacity same as if there is no interference
  - Accomplished by cleverly distributing the writing (codewords) and coloring their ink
  - Decoder must know how to read these codewords
Modulo Encoding/Decoding

- Received signal \( Y = X + S, \ -1 \leq X \leq 1 \)
  - \( S \) known to transmitter, not receiver
- Modulo operation removes the interference effects
  - Set \( X \) so that \( \lfloor Y \rfloor_{[-1,1]} = \) desired message (e.g. 0.5)
  - Receiver demodulates modulo \([-1,1]\)
Capacity Results

- Non-degraded broadcast channel
  - Receivers not necessarily “better” or “worse” due to multiple transmit/receive antennas
  - Capacity region for general case unknown

- Pioneering work by Caire/Shamai (Allerton’00):
  - Two TX antennas/two RXs (1 antenna each)
  - Dirty paper coding/lattice precoding (achievable rate)
    - Computationally very complex
  - MIMO version of the Sato upper bound
  - Upper bound is achievable: capacity known!
Dirty-Paper Coding (DPC) for MIMO BC

- Coding scheme:
  - Choose a codeword for user 1
  - Treat this codeword as interference to user 2
  - Pick signal for User 2 using “pre-coding”

- Receiver 2 experiences no interference:

  \[ R_2 = \log(\det(I + H_2 \Sigma_2 H_2^T)) \]

- Signal for Receiver 2 interferes with Receiver 1:

  \[ R_1 = \log\left( \frac{\det(I + H_1 (\Sigma_1 + \Sigma_2) H_1^T)}{\det(I + H_1 \Sigma_2 H_1^T)} \right) \]

- Encoding order can be switched

- DPC optimization highly complex
Does DPC achieve capacity?

- DPC yields MIMO BC achievable region.
  - We call this the dirty-paper region

- Is this region the capacity region?

- We use duality, dirty paper coding, and Sato’s upper bound to address this question

- First we need MIMO MAC Capacity
MIMO MAC Capacity

- MIMO MAC follows from MAC capacity formula

\[
C_{MAC}(P_1, \ldots, P_K) = \bigcup \left\{ (R_1, \ldots, R_K) : \sum_{k \in S} R_k \leq \log_2 \det \left[ I + \sum_{k \in S} H_k Q_k H_k^H \right] \right\},
\]

\[\forall S \subseteq \{1, \ldots, K\}\]

- Basic idea same as single user case
  - Pick some subset of users
  - The sum of those user rates equals the capacity as if the users pooled their power

- Power Allocation and Decoding Order
  - Each user has its own power (no power alloc.)
  - Decoding order depends on desired rate point
MIMO MAC with sum power

- **MAC with sum power:**
  - Transmitters code independently
  - Share power

\[
C_{MAC}^{Sum}(P) = \bigcup_{0 \leq P_1 \leq P} C_{MAC}(P_1, P - P_1)
\]

- **Theorem:** Dirty-paper BC region equals the dual sum-power MAC region

\[
C_{BC}^{DPC}(P) = C_{MAC}^{Sum}(P)
\]
Transformations: MAC to BC

- Show any rate achievable in sum-power MAC also achievable with DPC for BC:

\[ C_{BC}^{DPC}(P) \supseteq C_{MAC}^{Sum}(P) \]

- A sum-power MAC strategy for point \((R_1, \ldots, R_N)\) has a given input covariance matrix and encoding order.

- We find the corresponding PSD covariance matrix and encoding order to achieve \((R_1, \ldots, R_N)\) with DPC on BC:
  - The rank-preserving transform “flips the effective channel” and reverses the order.
  - Side result: beamforming is optimal for BC with 1 Rx antenna at each mobile.
Transformations: BC to MAC

- Show any rate achievable with DPC in BC also achievable in sum-power MAC:

$$C_{BC}^{DPC}(P) \subseteq C_{MAC}^{Sum}(P)$$

- We find transformation between optimal DPC strategy and optimal sum-power MAC strategy
  - “Flip the effective channel” and reverse order
Computing the Capacity Region

- Hard to compute DPC region (Caire/Shamai’00)
- “Easy” to compute the MIMO MAC capacity region
  - Obtain DPC region by solving for sum-power MAC and applying the theorem
  - Fast iterative algorithms have been developed
  - Greatly simplifies calculation of the DPC region and the associated transmit strategy

\[ C_{BC}^{DPC} (P) = C_{MAC}^{Sum} (P) \]
Sato Upper Bound on the BC Capacity Region

- Based on receiver cooperation

- BC sum rate capacity \( \leq \) Cooperative capacity

\[
C_{BC}^{\text{sum rate}}(P,H) \leq \max_{\Sigma_x} \frac{1}{2} \log \left| I + H \Sigma_x H^T \right|
\]
The Sato Bound for MIMO BC

- Introduce noise correlation between receivers
- BC capacity region unaffected
  - Only depends on noise marginals
- Tight Bound (Caire/Shamai’00)
  - Cooperative capacity with worst-case noise correlation
    \[
    C_{BC}^{\text{sumrate}}(P,H) \leq \inf \max \frac{1}{2} \log \left| I + \sum_z \Sigma_z^{-1/2} H \Sigma_x H^T \Sigma_z^{-1/2} \right|
    \]
- Explicit formula for worst-case noise covariance
- By Lagrangian duality, cooperative BC region equals the sum-rate capacity region of MIMO MAC
MIMO BC Capacity Bounds

Dirty Paper Achievable Region

BC Sum Rate Point

Sato Upper Bound

Does the DPC region equal the capacity region?
Full Capacity Region

- DPC gives us an achievable region
- Sato bound only touches at sum-rate point
- Bergman’s entropy power inequality is not a tight upper bound for nondegraded broadcast channel
- A tighter bound was needed to prove DPC optimal
  - It had been shown that if Gaussian codes optimal, DPC was optimal, but proving Gaussian optimality was open.
- Breakthrough by Weingarten, Steinberg and Shamai
  - Introduce notion of enhanced channel, applied Bergman’s converse to it to prove DPC optimal for MIMO BC.
Enhanced Channel Idea

- The aligned and degraded BC (AMBC)
  - Unity matrix channel, noise innovations process
  - Limit of AMBC capacity equals that of MIMO BC
  - Eigenvalues of some noise covariances go to infinity
  - Total power mapped to covariance matrix constraint

- Capacity region of AMBC achieved by Gaussian superposition coding and successive decoding
  - Uses entropy power inequality on enhanced channel
  - Enhanced channel has less noise variance than original
  - Can show that a power allocation exists whereby the enhanced channel rate is inside original capacity region

- By appropriate power alignment, capacities equal
Gaussian rate region of a two user 2 × 2 ADBC

$\mathcal{R}^G(S, N_{1\ldots m}^{''})$, tangential to $\mathcal{R}^G(S, N_{1\ldots m})$ at $R_1 = 1$

$\mathcal{R}^G(S, N_{1\ldots m}')$, departs from $\mathcal{R}^G(S, N_{1\ldots m})$ at $R_1 = 0.7$

Original

Enhanced
Main Points

- Shannon capacity gives fundamental data rate limits for multiuser wireless channels

- Fading multiuser channels optimize at each channel instance for maximum average rate

- Outage capacity has higher (fixed) rates than with no outage.

- OFDM is near optimal for broadcast channels with ISI

- Duality connects BC and MAC channels
  - Used to obtain capacity of one from the other

- Capacity of broadcast MIMO channel obtained using duality and the notion of an enhanced channel