Lecture 1

Linear quadratic regulator: Discrete-time finite horizon

- LQR cost function
- multi-objective interpretation
- LQR via least-squares
- dynamic programming solution
- steady-state LQR control
- extensions: time-varying systems, tracking problems

LQR problem: background

discrete-time system $x(t+1) = Ax(t) + Bu(t)$, $x(0) = x_0$

problem: choose $u(0), u(1), \ldots$ so that

- $x(0), x(1), \ldots$ is 'small', $i.e.$, we get good *regulation* or *control*
- $u(0), u(1), \ldots$ is 'small', $i.e.$, using small *input effort* or *actuator authority*
- we'll define 'small' soon
- these are usually competing objectives, $e.g.$, a large u can drive x to zero fast

linear quadratic regulator (LQR) theory addresses this question

LQR cost function

we define *quadratic cost function*

$$J(U) = \sum_{\tau=0}^{N-1} \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) + x(N)^T Q_f x(N)$$

where $U = (u(0), \dots, u(N-1))$ and

$$Q = Q^T \geq 0, \qquad Q_f = Q_f^T \geq 0, \qquad R = R^T > 0$$

are given *state cost*, *final state cost*, and *input cost* matrices

- N is called *time horizon* (we'll consider $N = \infty$ later)
- first term measures *state deviation*
- second term measures *input size* or *actuator authority*
- last term measures *final state deviation*
- Q, R set relative weights of state deviation and input usage
- $R > 0$ means any (nonzero) input adds to cost J

LQR problem: find $u_{\mathrm{lqr}}(0), \ldots, u_{\mathrm{lqr}}(N-1)$ that minimizes $J(U)$

Comparison to least-norm input

c.f. least-norm input that steers x to $x(N) = 0$:

- no cost attached to $x(0), \ldots, x(N-1)$
- $x(N)$ must be exactly zero

we can approximate the least-norm input by taking

$$R = I, \qquad Q = 0, \qquad Q_f \text{ large, } e.g., \ Q_f = 10^8 I$$

Multi-objective interpretation

common form for Q and R:

$$R = \rho I, \qquad Q = Q_f = C^T C$$

where $C \in \mathbf{R}^{p \times n}$ and $\rho \in \mathbf{R}$, $\rho > 0$

cost is then

$$J(U) = \sum_{\tau=0}^{N} \|y(\tau)\|^2 + \rho \sum_{\tau=0}^{N-1} \|u(\tau)\|^2$$

where $y = Cx$

here $\sqrt{\rho}$ gives relative weighting of output norm and input norm

Input and output objectives

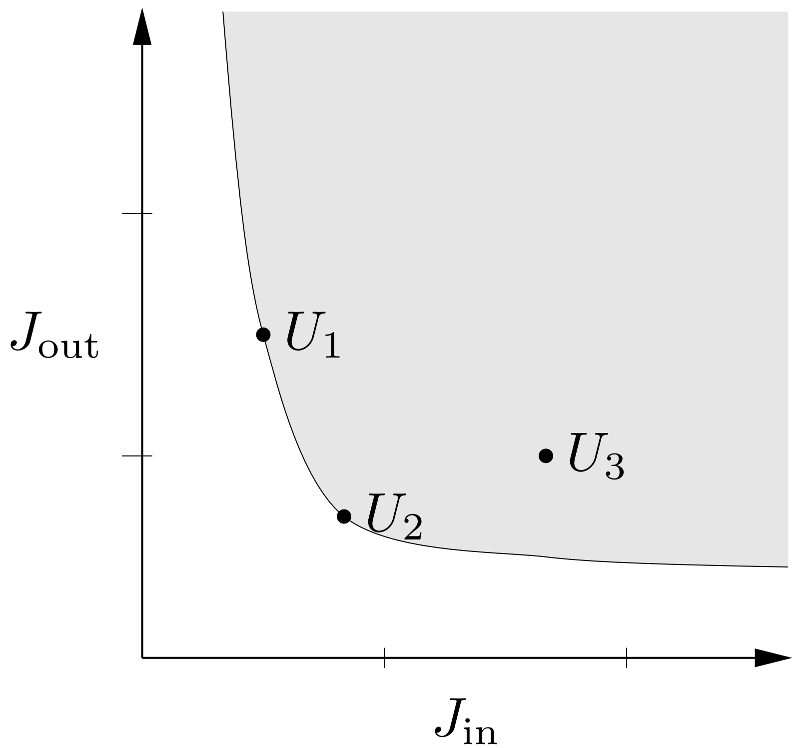
fix $x(0) = x_0$ and horizon N; for any input $U = (u(0), \ldots, u(N-1))$ define

- input cost $J_{\rm in}(U) = \sum_{\tau=0}^{N-1} \|u(\tau)\|^2$
- output cost $J_{\rm out}(U) = \sum_{\tau=0}^{N} \|y(\tau)\|^2$

these are (competing) objectives; we want *both* small

LQR quadratic cost is $J_{\rm out} + \rho J_{\rm in}$

plot $(J_{\text{in}}, J_{\text{out}})$ for all possible U:



- shaded area shows $(J_{\text{in}}, J_{\text{out}})$ achieved by some U
- clear area shows $(J_{\text{in}}, J_{\text{out}})$ not achieved by any U

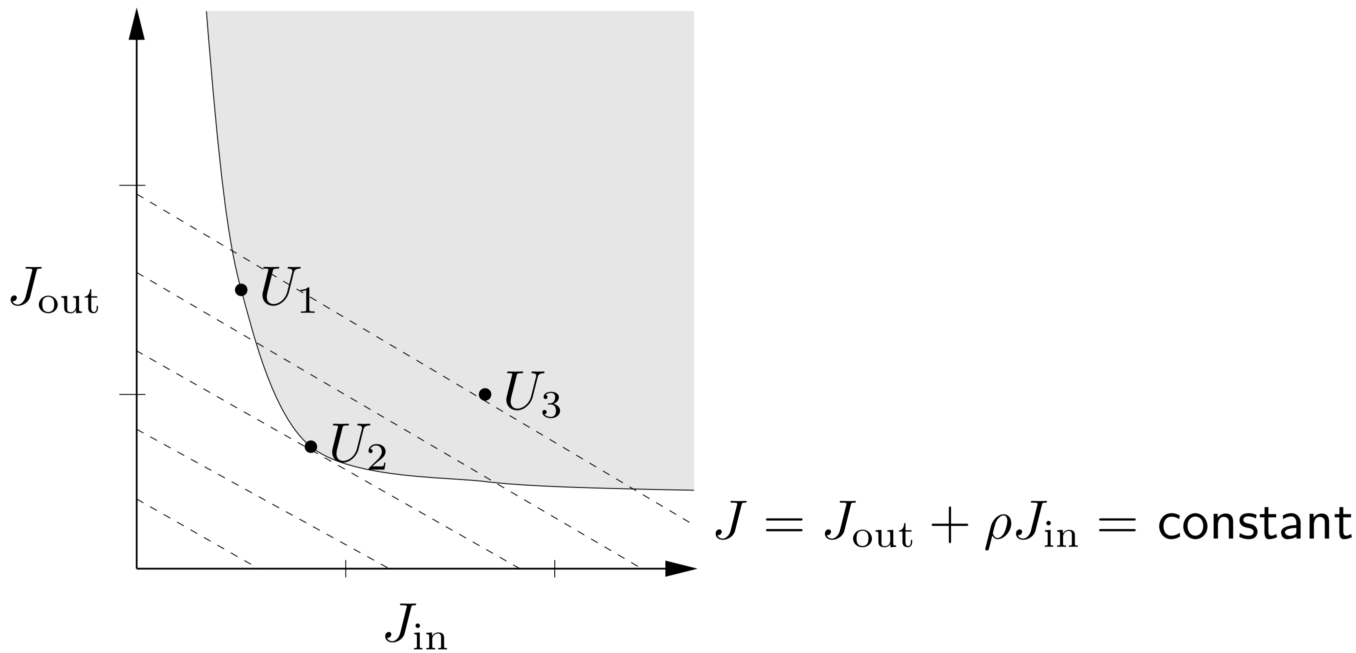
three sample inputs U_1, U_2, and U_3 are shown

- U_3 is worse than U_2 on both counts ($J_{\rm in}$ and $J_{\rm out}$)
- U_1 is better than U_2 in $J_{\rm in}$, but worse in $J_{\rm out}$

interpretation of LQR quadratic cost:

$$J = J_{\rm out} + \rho J_{\rm in} = \text{ constant}$$

corresponds to a line with slope $-\rho$ on $(J_{\rm in}, J_{\rm out})$ plot



- LQR optimal input is at boundary of shaded region, just touching line of smallest possible J

- u_2 is LQR optimal for ρ shown

- by varying ρ from 0 to $+\infty$, can sweep out *optimal tradeoff curve*

LQR via least-squares

LQR can be formulated (and solved) as a least-squares problem

$X = (x(0), \dots x(N))$ is a *linear function* of $x(0)$ and $U = (u(0), \dots, u(N-1))$:

$$\begin{bmatrix} x(0) \\ \vdots \\ x(N) \end{bmatrix} = \begin{bmatrix} 0 & \cdots & & \\ B & 0 & \cdots & \\ AB & B & 0 & \cdots \\ \vdots & \vdots & & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \begin{bmatrix} u(0) \\ \vdots \\ u(N-1) \end{bmatrix} + \begin{bmatrix} I \\ A \\ \vdots \\ A^N \end{bmatrix} x(0)$$

express as $X = GU + Hx(0)$, where $G \in \mathbf{R}^{Nn \times Nm}$, $H \in \mathbf{R}^{Nn \times n}$

express LQR cost as

$$\begin{aligned} J(U) &= \left\| \mathbf{diag}(Q^{1/2}, \ldots, Q^{1/2}, Q_f^{1/2})(GU + Hx(0)) \right\|^2 \\ &+ \left\| \mathbf{diag}(R^{1/2}, \ldots, R^{1/2})U \right\|^2 \end{aligned}$$

this is just a (big) least-squares problem

this solution method requires forming and solving a least-squares problem with size $N(n+m) \times Nm$

using a naive method ($e.g.$, QR factorization), cost is $O(N^3 nm^2)$

Dynamic programming solution

- gives an efficient, recursive method to solve LQR least-squares problem; cost is $O(Nn^3)$
- (but in fact, a less naive approach to solve the LQR least-squares problem will have the same complexity)
- useful and important idea on its own
- same ideas can be used for many other problems

Value function

for $t = 0, \ldots, N$ define the **value function** $V_t : \mathbf{R}^n \to \mathbf{R}$ by

$$V_t(z) = \min_{u(t),\ldots,u(N-1)} \sum_{\tau=t}^{N-1} \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) + x(N)^T Q_f x(N)$$

subject to $x(t) = z$, $x(\tau+1) = Ax(\tau) + Bu(\tau)$, $\tau = t, \ldots, T$

- $V_t(z)$ gives the minimum LQR cost-to-go, starting from state z at time t
- $V_0(x_0)$ is min LQR cost (from state x_0 at time 0)

we will find that

- V_t is quadratic, *i.e.*, $V_t(z) = z^T P_t z$, where $P_t = P_t^T \geq 0$
- P_t can be found recursively, working backward from $t = N$
- the LQR optimal u is easily expressed in terms of P_t

cost-to-go with no time left is just final state cost:

$$V_N(z) = z^T Q_f z$$

thus we have $P_N = Q_f$

Dynamic programming principle

- now suppose we know $V_{t+1}(z)$
- what is the optimal choice for $u(t)$?
- choice of $u(t)$ affects
  - current cost incurred (through $u(t)^T R u(t)$)
  - where we land, $x(t+1)$ (hence, the min-cost-to-go from $x(t+1)$)
- **dynamic programming (DP) principle:**

$$V_t(z) = \min_w \left(z^T Q z + w^T R w + V_{t+1}(Az + Bw) \right)$$

  - $z^T Q z + w^T R w$ is cost incurred at time t if $u(t) = w$
  - $V_{t+1}(Az + Bw)$ is min cost-to-go from where you land at $t+1$

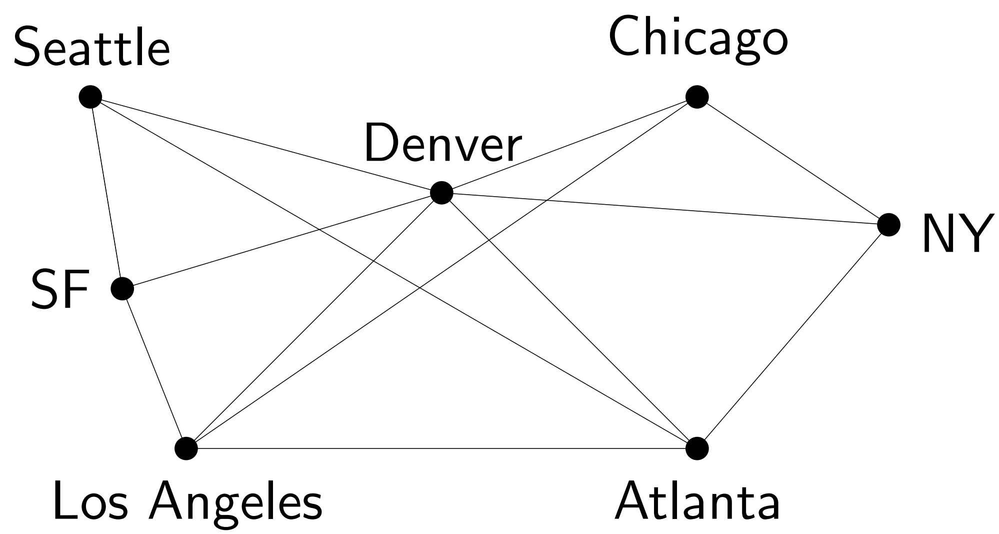
- follows from fact that we can minimize in any order:

$$\min_{w_1,\ldots,w_k} f(w_1,\ldots,w_k) = \min_{w_1} \underbrace{\left(\min_{w_2,\ldots,w_k} f(w_1,\ldots,w_k) \right)}_{\text{a fct of } w_1}$$

in words:
min cost-to-go from where you are = min over
(current cost incurred + min cost-to-go from where you land)

Example: path optimization

- edges show possible flights; each has some cost
- want to find min cost route or path from SF to NY



dynamic programming (DP):

- $V(i)$ is min cost from airport i to NY, over all possible paths
- to find min cost from city i to NY: minimize sum of flight cost plus min cost to NY from where you land, over all flights out of city i (gives optimal flight out of city i on way to NY)
- if we can find $V(i)$ for each i, we can find min cost path from any city to NY
- DP principle: $V(i) = \min_j (c_{ji} + V(j))$, where c_{ji} is cost of flight from i to j, and minimum is over all possible flights out of i

HJ equation for LQR

$$V_t(z) = z^T Q z + \min_w \left(w^T R w + V_{t+1}(Az + Bw) \right)$$

- called DP, Bellman, or Hamilton-Jacobi equation
- gives V_t recursively, in terms of V_{t+1}
- any minimizing w gives optimal $u(t)$:

$$u_{\mathrm{lqr}}(t) = \underset{w}{\operatorname{argmin}} \left(w^T R w + V_{t+1}(Az + Bw) \right)$$

- let's assume that $V_{t+1}(z) = z^T P_{t+1} z$, with $P_{t+1} = P_{t+1}^T \geq 0$

- we'll show that V_t has the same form

- by DP,

$$V_t(z) = z^T Q z + \min_w \left(w^T R w + (Az + Bw)^T P_{t+1}(Az + Bw) \right)$$

- can solve by setting derivative w.r.t. w to zero:

$$2w^T R + 2(Az + Bw)^T P_{t+1} B = 0$$

- hence optimal input is

$$w^* = -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A z$$

- and so (after some ugly algebra)

$$\begin{aligned} V_t(z) &= z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P_{t+1}(Az + Bw^*) \\ &= z^T \left(Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A\right) z \\ &= z^T P_t z \end{aligned}$$

where

$$P_t = Q + A^T P_{t+1} A - A^T P_{t+1} B (R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$$

- easy to show $P_t = P_t^T \geq 0$

Summary of LQR solution via DP

1. set $P_N := Q_f$

2. for $t = N, \ldots, 1$,

$$P_{t-1} := Q + A^T P_t A - A^T P_t B(R + B^T P_t B)^{-1} B^T P_t A$$

3. for $t = 0, \ldots, N-1$, define $K_t := -(R + B^T P_{t+1} B)^{-1} B^T P_{t+1} A$

4. for $t = 0, \ldots, N-1$, optimal u is given by $u_{\mathrm{lqr}}(t) = K_t x(t)$

- optimal u is a linear function of the state (called *linear state feedback*)
- recursion for min cost-to-go runs backward in time

LQR example

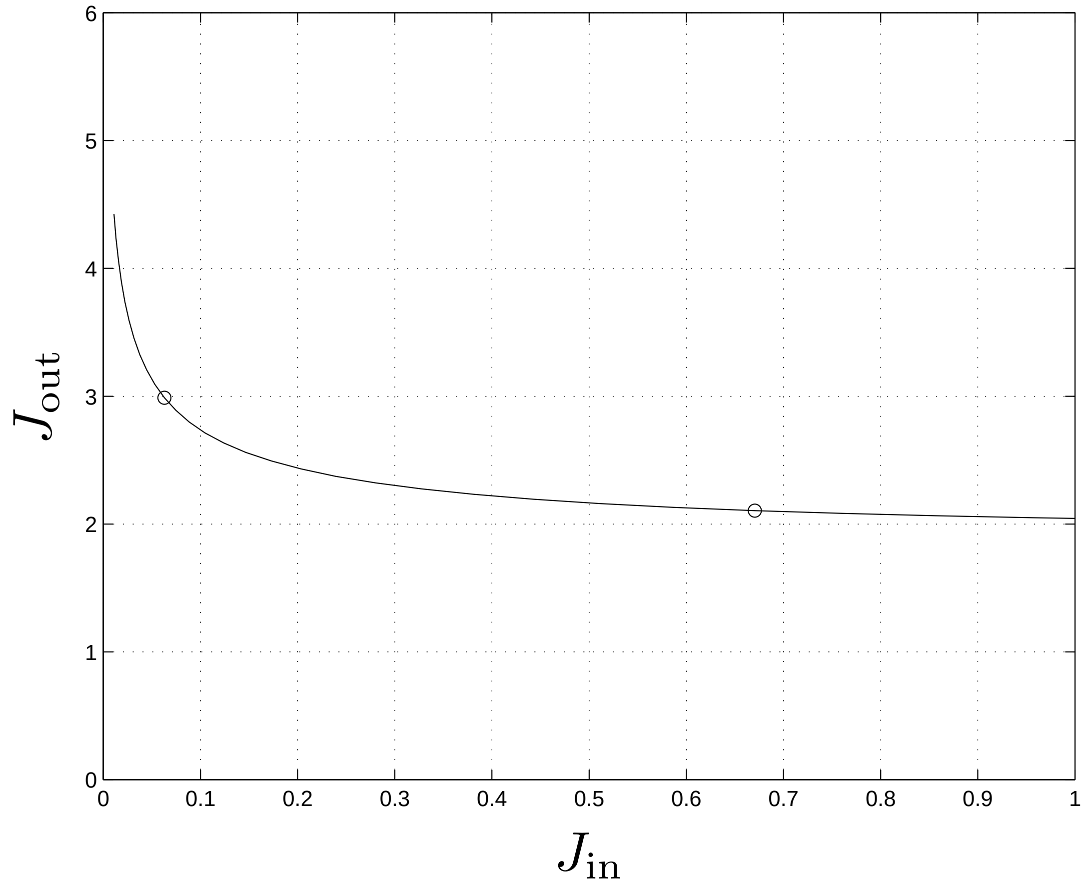
2-state, single-input, single-output system

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

with initial state $x(0) = (1, 0)$, horizon $N = 20$, and weight matrices

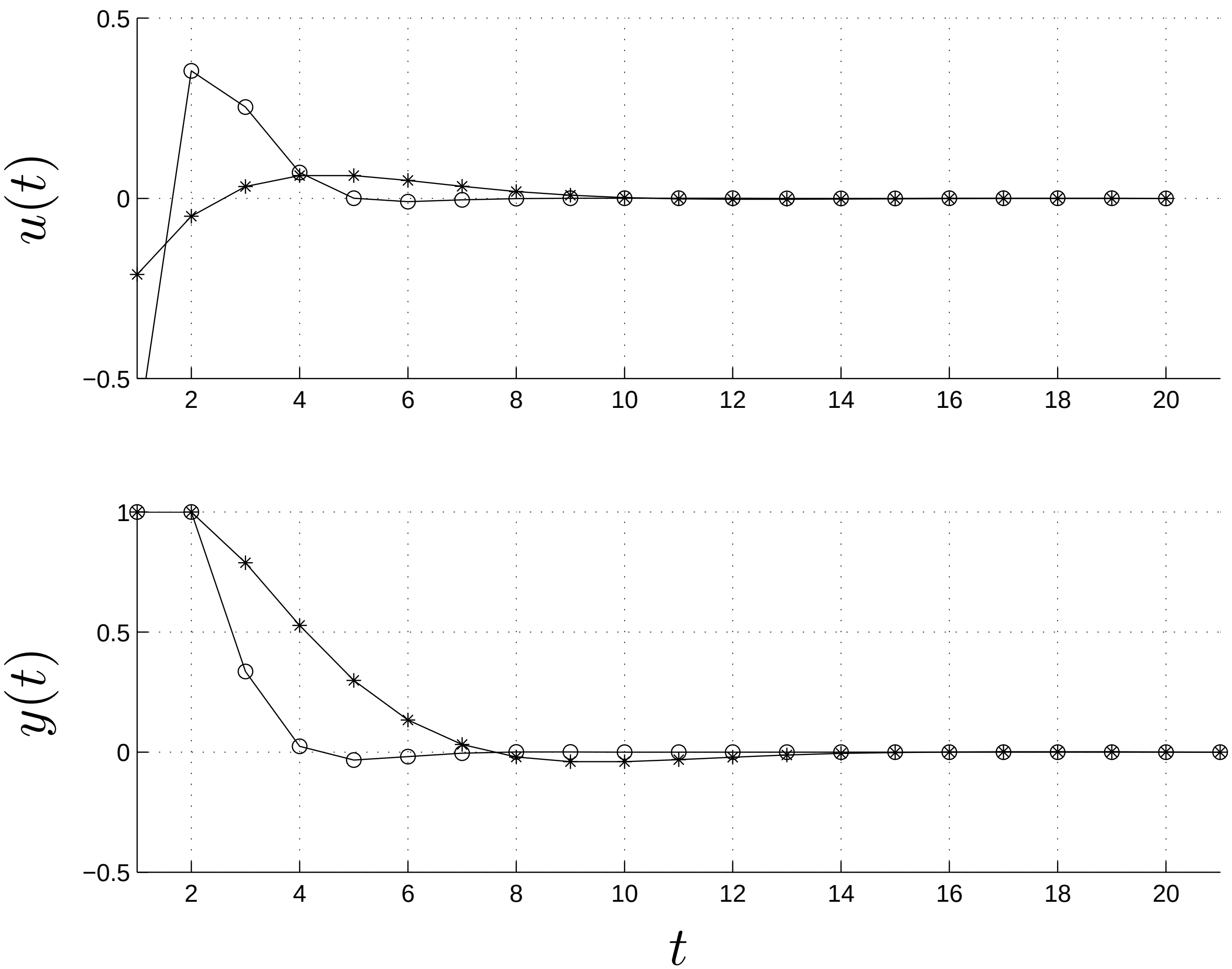
$$Q = Q_f = C^T C, \qquad R = \rho I$$

optimal trade-off curve of J_{in} vs. J_{out}:



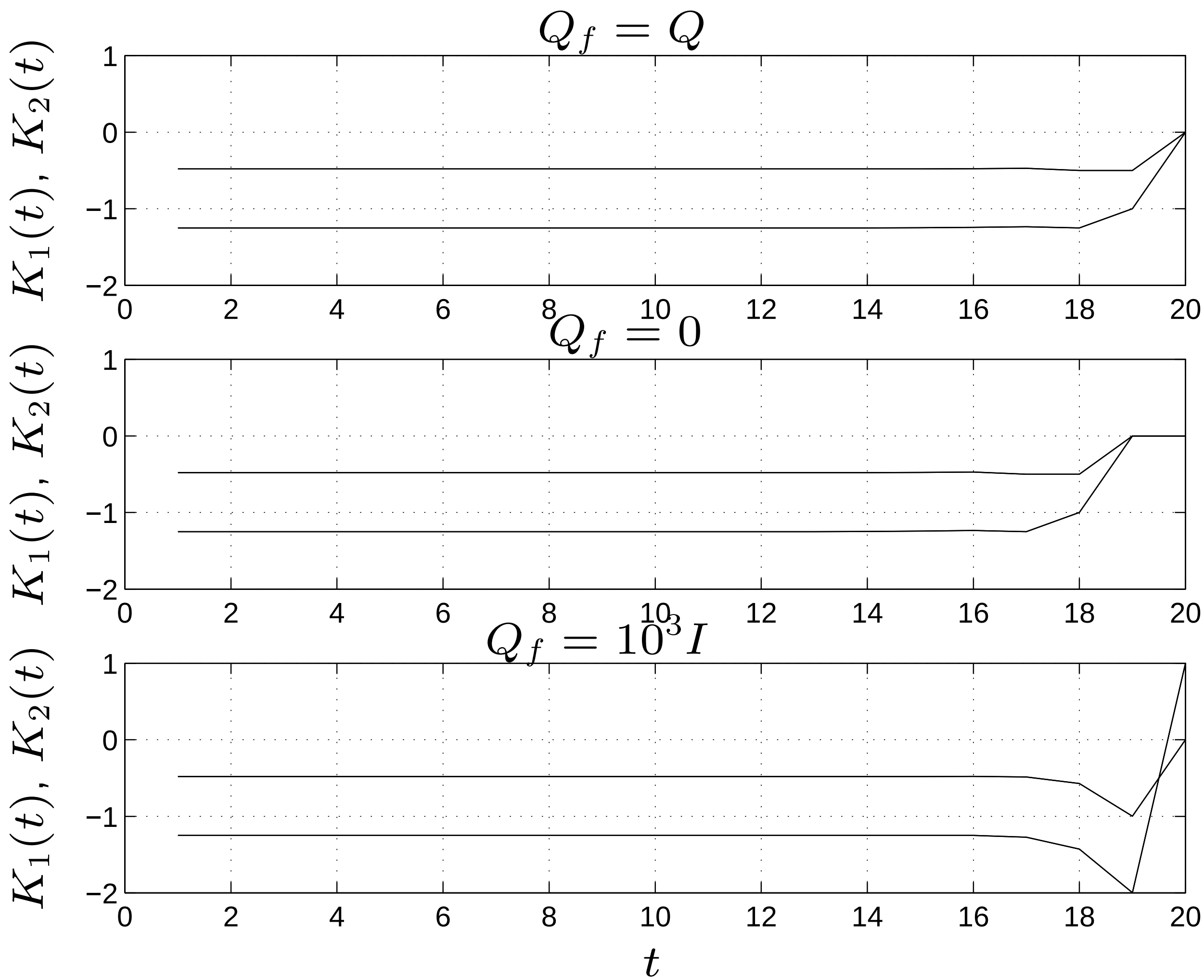
circles show LQR solutions with $\rho = 0.3$, $\rho = 10$

u & y for $\rho = 0.3$, $\rho = 10$:



optimal input has form $u(t) = K(t)x(t)$, where $K(t) \in \mathbf{R}^{1\times 2}$

state feedback gains vs. t for various values of Q_f (note convergence):



Steady-state regulator

usually P_t rapidly converges as t decreases below N

limit or steady-state value P_{ss} satisfies

$$P_{\mathrm{ss}} = Q + A^T P_{\mathrm{ss}} A - A^T P_{\mathrm{ss}} B (R + B^T P_{\mathrm{ss}} B)^{-1} B^T P_{\mathrm{ss}} A$$

which is called the (DT) algebraic Riccati equation (ARE)

- P_{ss} can be found by iterating the Riccati recursion, or by direct methods
- for t not close to horizon N, LQR optimal input is approximately a linear, constant state feedback

$$u(t) = K_{\mathrm{ss}} x(t), \qquad K_{\mathrm{ss}} = -(R + B^T P_{\mathrm{ss}} B)^{-1} B^T P_{\mathrm{ss}} A$$

(very widely used in practice; more on this later)

Time-varying systems

LQR is readily extended to handle time-varying systems

$$x(t+1) = A(t)x(t) + B(t)u(t)$$

and time-varying cost matrices

$$J = \sum_{\tau=0}^{N-1} \left(x(\tau)^T Q(\tau) x(\tau) + u(\tau)^T R(\tau) u(\tau) \right) + x(N)^T Q_f x(N)$$

(so Q_f is really just $Q(N)$)

DP solution is readily extended, but (of course) there need not be a steady-state solution

Tracking problems

we consider LQR cost with state and input offsets:

$$
\begin{aligned}
J &= \sum_{\tau=0}^{N-1} (x(\tau) - \bar{x}(\tau))^T Q (x(\tau) - \bar{x}(\tau)) \\
&+ \sum_{\tau=0}^{N-1} (u(\tau) - \bar{u}(\tau))^T R (u(\tau) - \bar{u}(\tau))
\end{aligned}
$$

(we drop the final state term for simplicity)

here, $\bar{x}(\tau)$ and $\bar{u}(\tau)$ are given desired state and input trajectories

DP solution is readily extended, even to time-varying tracking problems

Gauss-Newton LQR

nonlinear dynamical system: $x(t+1) = f(x(t), u(t))$, $x(0) = x_0$

objective is

$$J(U) = \sum_{\tau=0}^{N-1} \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right) + x(N)^T Q_f x(N)$$

where $Q = Q^T \geq 0$, $Q_f = Q_f^T \geq 0$, $R = R^T > 0$

start with a guess for U, and alternate between:

- linearize around current trajectory
- solve associated LQR (tracking) problem

sometimes converges, sometimes to the globally optimal U

some more detail:

- let u denote current iterate or guess
- simulate system to find x, using $x(t+1) = f(x(t), u(t))$
- linearize around this trajectory: $\delta x(t+1) = A(t)\delta x(t) + B(t)\delta u(t)$

$$A(t) = D_x f(x(t), u(t)) \qquad B(t) = D_u f(x(t), u(t))$$

- solve time-varying LQR tracking problem with cost

$$\begin{aligned} J &= \sum_{\tau=0}^{N-1} (x(\tau) + \delta x(\tau))^T Q (x(\tau) + \delta x(\tau)) \\ &+ \sum_{\tau=0}^{N-1} (u(\tau) + \delta u(\tau))^T R (u(\tau) + \delta u(\tau)) \end{aligned}$$

- for next iteration, set $u(t) := u(t) + \delta u(t)$

Lecture 2
LQR via Lagrange multipliers

- useful matrix identities
- linearly constrained optimization
- LQR via constrained optimization

Some useful matrix identities

let's start with a simple one:

$$Z(I+Z)^{-1} = I - (I+Z)^{-1}$$

(provided $I+Z$ is invertible)

to verify this identity, we start with

$$I = (I+Z)(I+Z)^{-1} = (I+Z)^{-1} + Z(I+Z)^{-1}$$

re-arrange terms to get identity

an identity that's a bit more complicated:

$$(I + XY)^{-1} = I - X(I + YX)^{-1}Y$$

(if either inverse exists, then the other does; in fact $\det(I + XY) = \det(I + YX)$)

to verify:

$$\begin{aligned}\left(I - X(I + YX)^{-1}Y\right)(I + XY) &= I + XY - X(I + YX)^{-1}Y(I + XY) \\ &= I + XY - X(I + YX)^{-1}(I + YX)Y \\ &= I + XY - XY = I\end{aligned}$$

another identity:

$$Y(I + XY)^{-1} = (I + YX)^{-1}Y$$

to verify this one, start with $Y(I + XY) = (I + YX)Y$

then multiply on left by $(I + YX)^{-1}$, on right by $(I + XY)^{-1}$

- note dimensions of inverses not necessarily the same
- mnemonic: lefthand Y moves into inverse, pushes righthand Y out . . .

and one more:

$$(I + XZ^{-1}Y)^{-1} = I - X(Z + YX)^{-1}Y$$

let's check:

$$\begin{aligned}
\left(I + X(Z^{-1}Y)\right)^{-1} &= I - X\left(I + Z^{-1}YX\right)^{-1}Z^{-1}Y \\
&= I - X(Z(I + Z^{-1}YX))^{-1}Y \\
&= I - X(Z + YX)^{-1}Y
\end{aligned}$$

Example: rank one update

- suppose we've already calculated or know A^{-1}, where $A \in \mathbf{R}^{n \times n}$
- we need to calculate $(A + bc^T)^{-1}$, where $b,\ c \in \mathbf{R}^n$
  ($A + bc^T$ is called a *rank one update* of A)

we'll use another identity, called *matrix inversion lemma*:

$$(A + bc^T)^{-1} = A^{-1} - \frac{1}{1 + c^T A^{-1} b}(A^{-1}b)(c^T A^{-1})$$

note that RHS is easy to calculate since we know A^{-1}

more general form of matrix inversion lemma:

$$(A+BC)^{-1} = A^{-1} - A^{-1}B\left(I + CA^{-1}B\right)^{-1} CA^{-1}$$

let's verify it:

$$\begin{aligned}
(A+BC)^{-1} &= \left(A(I + A^{-1}BC)\right)^{-1} \\
&= (I + (A^{-1}B)C)^{-1}A^{-1} \\
&= \left(I - (A^{-1}B)(I + C(A^{-1}B))^{-1}C\right) A^{-1} \\
&= A^{-1} - A^{-1}B(I + CA^{-1}B)^{-1}CA^{-1}
\end{aligned}$$

Another formula for the Riccati recursion

$$\begin{aligned}
P_{t-1} &= Q + A^T P_t A - A^T P_t B(R + B^T P_t B)^{-1} B^T P_t A \\
&= Q + A^T P_t \left(I - B(R + B^T P_t B)^{-1} B^T P_t\right) A \\
&= Q + A^T P_t \left(I - B((I + B^T P_t B R^{-1})R)^{-1} B^T P_t\right) A \\
&= Q + A^T P_t \left(I - B R^{-1}(I + B^T P_t B R^{-1})^{-1} B^T P_t\right) A \\
&= Q + A^T P_t \left(I + B R^{-1} B^T P_t\right)^{-1} A \\
&= Q + A^T \left(I + P_t B R^{-1} B^T\right)^{-1} P_t A
\end{aligned}$$

or, in pretty, symmetric form:

$$P_{t-1} = Q + A^T P_t^{1/2} \left(I + P_t^{1/2} B R^{-1} B^T P_t^{1/2}\right)^{-1} P_t^{1/2} A$$

Linearly constrained optimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Fx = g \end{array}$$

- $f : \mathbf{R}^n \to \mathbf{R}$ is smooth *objective function*
- $F \in \mathbf{R}^{m \times n}$ is fat

form *Lagrangian* $L(x, \lambda) = f(x) + \lambda^T(g - Fx)$ (λ is *Lagrange multiplier*)

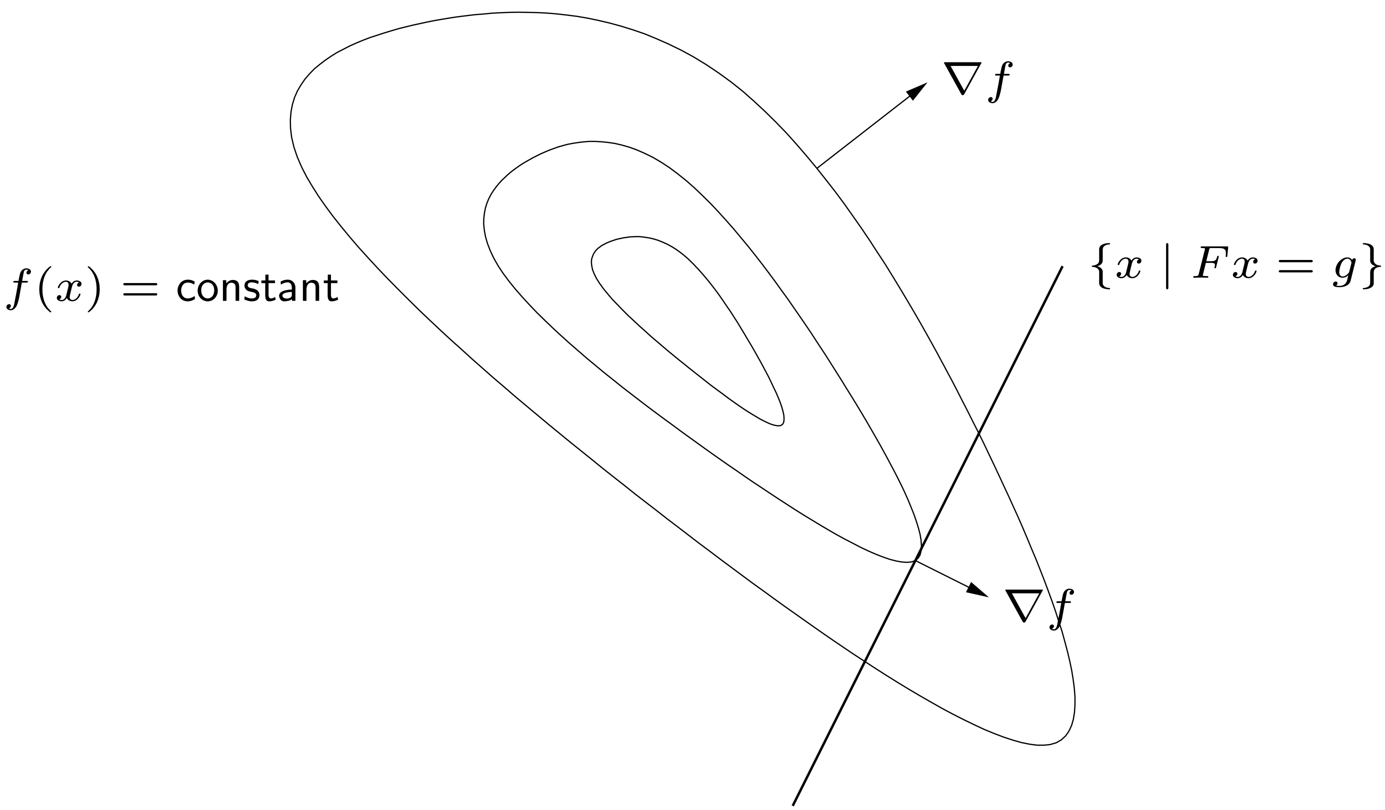
if x is optimal, then

$$\nabla_x L = \nabla f(x) - F^T \lambda = 0, \qquad \nabla_\lambda L = g - Fx = 0$$

$i.e.$, $\nabla f(x) = F^T \lambda$ for some $\lambda \in \mathbf{R}^m$

(generalizes optimality condition $\nabla f(x) = 0$ for unconstrained minimization problem)

Picture



$$\nabla f(x) = F^T \lambda \text{ for some } \lambda \iff \nabla f(x) \in \mathcal{R}(F^T) \iff \nabla f(x) \perp \mathcal{N}(F)$$

Feasible descent direction

suppose x is current, feasible point (*i.e.*, $Fx = g$)

consider a small step in direction v, to $x + hv$ (h small, positive)

when is $x + hv$ better than x?

need $x + hv$ feasible: $F(x + hv) = g + hFv = g$, so $Fv = 0$

$v \in \mathcal{N}(F)$ is called a *feasible direction*

we need $x + hv$ to have smaller objective than x:

$$f(x + hv) \approx f(x) + h\nabla f(x)^T v < f(x)$$

so we need $\nabla f(x)^T v < 0$ (called a *descent direction*)

(if $\nabla f(x)^T v > 0$, $-v$ is a descent direction, so we need only $\nabla f(x)^T v \neq 0$)

x is not optimal if there exists a feasible descent direction if x is optimal, every feasible direction satisfies $\nabla f(x)^T v = 0$

$$
\begin{aligned}
Fv = 0 \;\Rightarrow\; \nabla f(x)^T v = 0 \quad &\Longleftrightarrow \quad \mathcal{N}(F) \subseteq \mathcal{N}(\nabla f(x)^T) \\
&\Longleftrightarrow \quad \mathcal{R}(F^T) \supseteq \mathcal{R}(\nabla f(x)) \\
&\Longleftrightarrow \quad \nabla f(x) \in \mathcal{R}(F^T) \\
&\Longleftrightarrow \quad \nabla f(x) = F^T \lambda \;\; \text{for some} \;\; \lambda \in \mathbf{R}^m \\
&\Longleftrightarrow \quad \nabla f(x) \perp \mathcal{N}(F)
\end{aligned}
$$

LQR as constrained minimization problem

minimize	$J = \frac{1}{2}\sum_{t=0}^{N-1}\big(x(t)^TQx(t) + u(t)^TRu(t)\big) + \frac{1}{2}x(N)^TQ_fx(N)$
subject to	$x(t+1) = Ax(t) + Bu(t), \quad t = 0, \ldots, N-1$

- variables are $u(0), \ldots, u(N-1)$ and $x(1), \ldots, x(N)$ ($x(0) = x_0$ is given)
- objective is (convex) quadratic (factor $1/2$ in objective is for convenience)

introduce Lagrange multipliers $\lambda(1), \ldots, \lambda(N) \in \mathbf{R}^n$ and form Lagrangian

$$L = J + \sum_{t=0}^{N-1} \lambda(t+1)^T \left(Ax(t) + Bu(t) - x(t+1)\right)$$

Optimality conditions

we have $x(t+1) = Ax(t) + Bu(t)$ for $t = 0, \ldots, N-1$, $x(0) = x_0$

for $t = 0, \ldots, N-1$, $\nabla_{u(t)} L = Ru(t) + B^T \lambda(t+1) = 0$

hence, $u(t) = -R^{-1} B^T \lambda(t+1)$

for $t = 1, \ldots, N-1$, $\nabla_{x(t)} L = Qx(t) + A^T \lambda(t+1) - \lambda(t) = 0$

hence, $\lambda(t) = A^T \lambda(t+1) + Qx(t)$

$\nabla_{x(N)} L = Q_f x(N) - \lambda(N) = 0$, so $\lambda(N) = Q_f x(N)$

these are a set of linear equations in the variables

$$u(0), \ldots, u(N-1), \quad x(1), \ldots, x(N), \quad \lambda(1), \ldots, \lambda(N)$$

Co-state equations

optimality conditions break into two parts:

$$x(t+1) = Ax(t) + Bu(t), \qquad x(0) = x_0$$

this recursion for state x runs forward in time, with initial condition

$$\lambda(t) = A^T\lambda(t+1) + Qx(t), \qquad \lambda(N) = Q_f x(N)$$

this recursion for λ runs backward in time, with final condition

- λ is called *co-state*
- recursion for λ sometimes called *adjoint system*

Solution via Riccati recursion

we will see that $\lambda(t) = P_t x(t)$, where P_t is the min-cost-to-go matrix defined by the Riccati recursion

thus, Riccati recursion gives clever way to solve this set of linear equations

it holds for $t = N$, since $P_N = Q_f$ and $\lambda(N) = Q_f x(N)$

now suppose it holds for $t+1$, $i.e.$, $\lambda(t+1) = P_{t+1}x(t+1)$

let's show it holds for t, $i.e.$, $\lambda(t) = P_t x(t)$

using $x(t+1) = Ax(t) + Bu(t)$ and $u(t) = -R^{-1}B^T\lambda(t+1)$,

$$\lambda(t+1) = P_{t+1}(Ax(t) + Bu(t)) = P_{t+1}(Ax(t) - BR^{-1}B^T\lambda(t+1))$$

so

$$\lambda(t+1) = (I + P_{t+1}BR^{-1}B^T)^{-1}P_{t+1}Ax(t)$$

using $\lambda(t) = A^T\lambda(t+1) + Qx(t)$, we get

$$\lambda(t) = A^T(I + P_{t+1}BR^{-1}B^T)^{-1}P_{t+1}Ax(t) + Qx(t) = P_t x(t)$$

since by the Riccati recursion

$$P_t = Q + A^T(I + P_{t+1}BR^{-1}B^T)^{-1}P_{t+1}A$$

this proves $\lambda(t) = P_t x(t)$

let's check that our two formulas for $u(t)$ are consistent:

$$\begin{aligned} u(t) &= -R^{-1}B^T\lambda(t+1) \\ &= -R^{-1}B^T(I+P_{t+1}BR^{-1}B^T)^{-1}P_{t+1}Ax(t) \\ &= -R^{-1}(I+B^TP_{t+1}BR^{-1})^{-1}B^TP_{t+1}Ax(t) \\ &= -((I+B^TP_{t+1}BR^{-1})R)^{-1}B^TP_{t+1}Ax(t) \\ &= -(R+B^TP_{t+1}B)^{-1}B^TP_{t+1}Ax(t) \end{aligned}$$

which is what we had before

Lecture 3
Infinite horizon linear quadratic regulator

- infinite horizon LQR problem
- dynamic programming solution
- receding horizon LQR control
- closed-loop system

Infinite horizon LQR problem

discrete-time system $x(t+1) = Ax(t) + Bu(t)$, $x(0) = x_0$

problem: choose $u(0), u(1), \ldots$ to minimize

$$J = \sum_{\tau=0}^{\infty} \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right)$$

with given constant state and input weight matrices

$$Q = Q^T \geq 0, \qquad R = R^T > 0$$

. . . an *infinite dimensional problem*

problem: it's possible that $J = \infty$ for all input sequences $u(0), \ldots$

$$x(t+1) = 2x(t) + 0u(t), \qquad x(0) = 1$$

let's assume (A, B) is controllable

then for any x_0 there's an input sequence

$$u(0), \ldots, u(n-1),\ 0,\ 0, \ldots$$

that steers x to zero at $t = n$, and keeps it there

for this u, $J < \infty$

and therefore, $\min_u J < \infty$ for any x_0

Dynamic programming solution

define **value function** $V : \mathbf{R}^n \to \mathbf{R}$

$$V(z) = \min_{u(0),\ldots} \sum_{\tau=0}^{\infty} \left(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau) \right)$$

subject to $x(0) = z$, $x(\tau+1) = Ax(\tau) + Bu(\tau)$

- $V(z)$ is the minimum LQR cost-to-go, starting from state z
- doesn't depend on time-to-go, which is always ∞; infinite horizon problem is *shift invariant*

Hamilton-Jacobi equation

fact: V is quadratic, *i.e.*, $V(z) = z^T P z$, where $P = P^T \geq 0$
(can be argued directly from first principles)

HJ equation:

$$V(z) = \min_w \left(z^T Q z + w^T R w + V(Az + Bw) \right)$$

or

$$z^T P z = \min_w \left(z^T Q z + w^T R w + (Az + Bw)^T P (Az + Bw) \right)$$

minimizing w is $w^* = -(R + B^T P B)^{-1} B^T P A z$

so HJ equation is

$$\begin{aligned} z^T P z &= z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P (Az + Bw^*) \\ &= z^T \left(Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A \right) z \end{aligned}$$

this must hold for all z, so we conclude that P satisfies the ARE

$$P = Q + A^T PA - A^T PB(R + B^T PB)^{-1} B^T PA$$

and the optimal input is constant state feedback $u(t) = Kx(t)$,

$$K = -(R + B^T PB)^{-1} B^T PA$$

compared to finite-horizon LQR problem,

- value function and optimal state feedback gains are time-invariant
- we don't have a recursion to compute P; we only have the ARE

fact: the ARE has only one positive semidefinite solution P

i.e., ARE plus $P = P^T \geq 0$ uniquely characterizes value function

consequence: the Riccati recursion

$$P_{k+1} = Q + A^T P_k A - A^T P_k B(R + B^T P_k B)^{-1} B^T P_k A, \qquad P_1 = Q$$

converges to the unique PSD solution of the ARE
(when (A, B) controllable)

(later we'll see direct methods to solve ARE)

thus, infinite-horizon LQR optimal control is same as steady-state finite horizon optimal control

Receding-horizon LQR control

consider cost function

$$J_t(u(t),\ldots,u(t+T-1)) = \sum_{\tau=t}^{\tau=t+T} \big(x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\big)$$

- T is called *horizon*
- same as infinite horizon LQR cost, truncated after T steps into future

if $(u(t)^*,\ldots,u(t+T-1)^*)$ minimizes J_t, $u(t)^*$ is called (T-step ahead) *optimal receding horizon control*

in words:

- at time t, find input sequence that minimizes T-step-ahead LQR cost, starting at current time
- then use only the first input

example: 1-step ahead receding horizon control

find $u(t),\ u(t+1)$ that minimize

$$J_t = x(t)^T Q x(t) + x(t+1)^T Q x(t+1) + u(t)^T R u(t) + u(t+1)^T R u(t+1)$$

first term doesn't matter; optimal choice for $u(t+1)$ is 0; optimal $u(t)$ minimizes

$$x(t{+}1)^T Q x(t{+}1) + u(t)^T R u(t) = (Ax(t){+}Bu(t))^T Q (Ax(t){+}Bu(t)) + u(t)^T R u(t)$$

thus, 1-step ahead receding horizon optimal input is

$$u(t) = -(R + B^T Q B)^{-1} B^T Q A x(t)$$

. . . a constant state feedback

in general, optimal T-step ahead LQR control is

$$u(t) = K_T x(t), \qquad K_T = -(R + B^T P_T B)^{-1} B^T P_T A$$

where

$$P_1 = Q, \qquad P_{i+1} = Q + A^T P_i A - A^T P_i B (R + B^T P_i B)^{-1} B^T P_i A$$

$i.e.$: same as the optimal finite horizon LQR control, $T - 1$ steps before the horizon N

- a constant state feedback
- state feedback gain converges to infinite horizon optimal as horizon becomes long (assuming controllability)

Closed-loop system

suppose K is LQR-optimal state feedback gain

$$x(t+1) = Ax(t) + Bu(t) = (A + BK)x(t)$$

is called *closed-loop system*

($x(t+1) = Ax(t)$ is called *open-loop system*)

is closed-loop system stable? consider

$$x(t+1) = 2x(t) + u(t), \qquad Q = 0, \qquad R = 1$$

optimal control is $u(t) = 0x(t)$, *i.e.*, closed-loop system is unstable

fact: if (Q, A) observable and (A, B) controllable, then closed-loop system is stable

Lecture 4
Continuous time linear quadratic regulator

- continuous-time LQR problem
- dynamic programming solution
- Hamiltonian system and two point boundary value problem
- infinite horizon LQR
- direct solution of ARE via Hamiltonian

Continuous-time LQR problem

continuous-time system $\dot{x} = Ax + Bu$, $x(0) = x_0$

problem: choose $u : \mathbf{R}_+ \to \mathbf{R}^m$ to minimize

$$J = \int_0^T x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau + x(T)^T Q_f x(T)$$

- T is *time horizon*
- $Q = Q^T \geq 0$, $Q_f = Q_f^T \geq 0$, $R = R^T > 0$ are *state cost*, *final state cost*, and *input cost* matrices

. . . an *infinite-dimensional problem*: (trajectory $u : \mathbf{R}_+ \to \mathbf{R}^m$ is variable)

Dynamic programming solution

we'll solve LQR problem using dynamic programming

for $0 \le t \le T$ we define the **value function** $V_t : \mathbf{R}^n \to \mathbf{R}$ by

$$V_t(z) = \min_u \int_t^T x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau + x(T)^T Q_f x(T)$$

subject to $x(t) = z$, $\dot{x} = Ax + Bu$

- minimum is taken over all possible signals $u : [t, T] \to \mathbf{R}^m$
- $V_t(z)$ gives the minimum LQR cost-to-go, starting from state z at time t
- $V_T(z) = z^T Q_f z$

fact: V_t is quadratic, $i.e.$, $V_t(z) = z^T P_t z$, where $P_t = P_t^T \geq 0$

similar to discrete-time case:

- P_t can be found from a *differential equation* running backward in time from $t = T$
- the LQR optimal u is easily expressed in terms of P_t

we start with $x(t) = z$

let's take $u(t) = w \in \mathbf{R}^m$, a constant, over the time interval $[t, t+h]$, where $h > 0$ is small

cost incurred over $[t, t+h]$ is

$$\int_t^{t+h} x(\tau)^T Q x(\tau) + w^T R w \, d\tau \approx h(z^T Q z + w^T R w)$$

and we end up at $x(t+h) \approx z + h(Az + Bw)$

min-cost-to-go from where we land is approximately

$$\begin{array}{rcl}
\multicolumn{3}{l}{V_{t+h}(z+h(Az+Bw))} \\
& = & (z+h(Az+Bw))^T P_{t+h}(z+h(Az+Bw)) \\
& \approx & (z+h(Az+Bw))^T (P_t + h\dot{P}_t)(z+h(Az+Bw)) \\
& \approx & z^T P_t z + h\left((Az+Bw)^T P_t z + z^T P_t(Az+Bw) + z^T \dot{P}_t z\right)
\end{array}$$

(dropping h^2 and higher terms)

cost incurred plus min-cost-to-go is approximately

$$z^T P_t z + h\left(z^T Q z + w^T R w + (Az+Bw)^T P_t z + z^T P_t(Az+Bw) + z^T \dot{P}_t z\right)$$

minimize over w to get (approximately) optimal w:

$$2hw^T R + 2hz^T P_t B = 0$$

so

$$w^* = -R^{-1}B^T P_t z$$

thus optimal u is time-varying linear state feedback:

$$u_{\mathrm{lqr}}(t) = K_t x(t), \qquad K_t = -R^{-1}B^T P_t$$

HJ equation

now let's substitute w^* into HJ equation:

$$z^T P_t z \approx z^T P_t z +$$
$$+ h\left(z^T Q z + w^{*T} R w^* + (Az + Bw^*)^T P_t z + z^T P_t (Az + Bw^*) + z^T \dot{P}_t z\right)$$

yields, after simplification,

$$-\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q$$

which is the *Riccati differential equation* for the LQR problem

we can solve it (numerically) using the *final condition* $P_T = Q_f$

Summary of cts-time LQR solution via DP

1. solve Riccati differential equation

$$-\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q, \qquad P_T = Q_f$$

(backward in time)

2. optimal u is $u_{\mathrm{lqr}}(t) = K_t x(t)$, $K_t := -R^{-1} B^T P_t$

DP method readily extends to time-varying A, B, Q, R, and tracking problem

Steady-state regulator

usually P_t rapidly converges as t decreases below T

limit $P_{\rm ss}$ satisfies (cts-time) algebraic Riccati equation (ARE)

$$A^TP + PA - PBR^{-1}B^TP + Q = 0$$

a quadratic matrix equation

- $P_{\rm ss}$ can be found by (numerically) integrating the Riccati differential equation, or by direct methods
- for t not close to horizon T, LQR optimal input is approximately a linear, constant state feedback

$$u(t) = K_{\rm ss}x(t), \qquad K_{\rm ss} = -R^{-1}B^TP_{\rm ss}$$

Derivation via discretization

let's discretize using small step size $h > 0$, with $Nh = T$

$$x((k+1)h) \approx x(kh) + h\dot{x}(kh) = (I + hA)x(kh) + hBu(kh)$$

$$J \approx \frac{h}{2}\sum_{k=0}^{N-1}\left(x(kh)^T Q x(kh) + u(kh)^T R u(kh)\right) + \frac{1}{2}x(Nh)^T Q_f x(Nh)$$

this yields a discrete-time LQR problem, with parameters

$$\tilde{A} = I + hA, \qquad \tilde{B} = hB, \qquad \tilde{Q} = hQ, \qquad \tilde{R} = hR, \qquad \tilde{Q}_f = Q_f$$

solution to discrete-time LQR problem is $u(kh) = \tilde{K}_k x(kh)$,

$$\tilde{K}_k = -(\tilde{R} + \tilde{B}^T \tilde{P}_{k+1} \tilde{B})^{-1} \tilde{B}^T \tilde{P}_{k+1} \tilde{A}$$

$$\tilde{P}_{k-1} = \tilde{Q} + \tilde{A}^T \tilde{P}_k \tilde{A} - \tilde{A}^T \tilde{P}_k \tilde{B} (\tilde{R} + \tilde{B}^T \tilde{P}_k \tilde{B})^{-1} \tilde{B}^T \tilde{P}_k \tilde{A}$$

substituting and keeping only h^0 and h^1 terms yields

$$\tilde{P}_{k-1} = hQ + \tilde{P}_k + hA^T \tilde{P}_k + h\tilde{P}_k A - h\tilde{P}_k B R^{-1} B^T \tilde{P}_k$$

which is the same as

$$-\frac{1}{h}(\tilde{P}_k - \tilde{P}_{k-1}) = Q + A^T \tilde{P}_k + \tilde{P}_k A - \tilde{P}_k B R^{-1} B^T \tilde{P}_k$$

letting $h \to 0$ we see that $\tilde{P}_k \to P_{kh}$, where

$$-\dot{P} = Q + A^T P + PA - PBR^{-1}B^T P$$

similarly, we have

$$\begin{aligned}
\tilde{K}_k &= -(\tilde{R} + \tilde{B}^T \tilde{P}_{k+1} \tilde{B})^{-1} \tilde{B}^T \tilde{P}_{k+1} \tilde{A} \\
&= -(hR + h^2 B^T \tilde{P}_{k+1} B)^{-1} h B^T \tilde{P}_{k+1} (I + hA) \\
&\to -R^{-1} B^T P_{kh}
\end{aligned}$$

as $h \to 0$

Derivation using Lagrange multipliers

pose as constrained problem:

$$\begin{array}{ll} \text{minimize} & J = \frac{1}{2}\int_0^T x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau + \frac{1}{2} x(T)^T Q_f x(T) \\ \text{subject to} & \dot{x}(t) = Ax(t) + Bu(t), \quad t \in [0,T] \end{array}$$

- optimization variable is *function* $u : [0,T] \to \mathbf{R}^m$
- infinite number of equality constraints, one for each $t \in [0,T]$

introduce Lagrange multiplier *function* $\lambda : [0,T] \to \mathbf{R}^n$ and form

$$L = J + \int_0^T \lambda(\tau)^T (Ax(\tau) + Bu(\tau) - \dot{x}(\tau))\, d\tau$$

Optimality conditions

(note: you need *distribution theory* to really make sense of the derivatives here . . .)

from $\nabla_{u(t)} L = Ru(t) + B^T \lambda(t) = 0$ we get $u(t) = -R^{-1} B^T \lambda(t)$

to find $\nabla_{x(t)} L$, we use

$$\int_0^T \lambda(\tau)^T \dot{x}(\tau)\, d\tau = \lambda(T)^T x(T) - \lambda(0)^T x(0) - \int_0^T \dot{\lambda}(\tau)^T x(\tau)\, d\tau$$

from $\nabla_{x(t)} L = Qx(t) + A^T \lambda(t) + \dot{\lambda}(t) = 0$ we get

$$\dot{\lambda}(t) = -A^T \lambda(t) - Qx(t)$$

from $\nabla_{x(T)} L = Q_f x(T) - \lambda(T) = 0$, we get $\lambda(T) = Q_f x(T)$

Co-state equations

optimality conditions are

$$\dot{x} = Ax + Bu, \quad x(0) = x_0, \qquad \dot{\lambda} = -A^T\lambda - Qx, \quad \lambda(T) = Q_f x(T)$$

using $u(t) = -R^{-1}B^T\lambda(t)$, can write as

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

- $2n \times 2n$ matrix above is called *Hamiltonian* for problem
- with conditions $x(0) = x_0$, $\lambda(T) = Q_f x(T)$, called *two-point boundary value problem*

as in discrete-time case, we can show that $\lambda(t) = P_t x(t)$, where

$$-\dot{P}_t = A^T P_t + P_t A - P_t B R^{-1} B^T P_t + Q, \qquad P_T = Q_f$$

in other words, value function P_t gives simple relation between x and λ

to show this, we show that $\lambda = Px$ satisfies co-state equation
$\dot{\lambda} = -A^T \lambda - Qx$

$$\begin{aligned}
\dot{\lambda} &= \frac{d}{dt}(Px) = \dot{P}x + P\dot{x} \\
&= -(Q + A^T P + PA - PBR^{-1}B^T P)x + P(Ax - BR^{-1}B^T \lambda) \\
&= -Qx - A^T Px + PBR^{-1}B^T Px - PBR^{-1}B^T Px \\
&= -Qx - A^T \lambda
\end{aligned}$$

Solving Riccati differential equation via Hamiltonian

the (quadratic) Riccati differential equation

$$-\dot{P} = A^T P + PA - PBR^{-1}B^T P + Q$$

and the (linear) Hamiltonian differential equation

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix}$$

are closely related

$\lambda(t) = P_t x(t)$ suggests that P should have the form $P_t = \lambda(t)x(t)^{-1}$
(but this doesn't make sense unless x and λ are scalars)

consider the Hamiltonian *matrix* (linear) differential equation

$$\frac{d}{dt}\begin{bmatrix} X(t) \\ Y(t) \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} X(t) \\ Y(t) \end{bmatrix}$$

where $X(t),\ Y(t) \in \mathbf{R}^{n\times n}$

then, $Z(t) = Y(t)X(t)^{-1}$ satisfies Riccati differential equation

$$-\dot{Z} = A^T Z + ZA - ZBR^{-1}B^T Z + Q$$

hence we can solve Riccati DE by solving (linear) matrix Hamiltonian DE, with final conditions $X(T) = I$, $Y(T) = Q_f$, and forming $P(t) = Y(t)X(t)^{-1}$

$$
\begin{aligned}
\dot{Z} &= \frac{d}{dt} Y X^{-1} \\
&= \dot{Y} X^{-1} - Y X^{-1} \dot{X} X^{-1} \\
&= (-QX - A^T Y) X^{-1} - Y X^{-1} \left(AX - BR^{-1} B^T Y \right) X^{-1} \\
&= -Q - A^T Z - ZA + ZBR^{-1} B^T Z
\end{aligned}
$$

where we use two identities:

- $\dfrac{d}{dt} (F(t)G(t)) = \dot{F}(t)G(t) + F(t)\dot{G}(t)$

- $\dfrac{d}{dt} \left(F(t)^{-1} \right) = -F(t)^{-1} \dot{F}(t) F(t)^{-1}$

Infinite horizon LQR

we now consider the infinite horizon cost function

$$J = \int_0^\infty x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau$$

we define the value function as

$$V(z) = \min_u \int_0^\infty x(\tau)^T Q x(\tau) + u(\tau)^T R u(\tau)\, d\tau$$

subject to $x(0) = z$, $\dot{x} = Ax + Bu$

we assume that (A, B) is controllable, so V is finite for all z

can show that V is quadratic: $V(z) = z^T P z$, where $P = P^T \geq 0$

optimal u is $u(t) = Kx(t)$, where $K = -R^{-1}B^TP$
($i.e.$, a constant linear state feedback)

HJ equation is ARE

$$Q + A^TP + PA - PBR^{-1}B^TP = 0$$

which together with $P \geq 0$ characterizes P

can solve as limiting value of Riccati DE, or via direct method

Closed-loop system

with K LQR optimal state feedback gain, closed-loop system is

$$\dot{x} = Ax + Bu = (A + BK)x$$

fact: closed-loop system is stable when (Q, A) observable and (A, B) controllable

we denote eigenvalues of $A + BK$, called *closed-loop eigenvalues*, as $\lambda_1, \ldots, \lambda_n$

with assumptions above, $\Re\lambda_i < 0$

Solving ARE via Hamiltonian

$$\begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} A - BR^{-1}B^T P \\ -Q - A^T P \end{bmatrix} = \begin{bmatrix} A + BK \\ -Q - A^T P \end{bmatrix}$$

and so

$$\begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} = \begin{bmatrix} A + BK & -BR^{-1}B^T \\ 0 & -(A+BK)^T \end{bmatrix}$$

where 0 in lower left corner comes from ARE

note that

$$\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$$

we see that:

- eigenvalues of Hamiltonian H are $\lambda_1, \ldots, \lambda_n$ and $-\lambda_1, \ldots, -\lambda_n$
- hence, closed-loop eigenvalues are the eigenvalues of H with negative real part

let's assume $A + BK$ is diagonalizable, $i.e.$,

$$T^{-1}(A + BK)T = \Lambda = \mathbf{diag}(\lambda_1, \ldots, \lambda_n)$$

then we have $T^T(-A - BK)^T T^{-T} = -\Lambda$, so

$$\begin{aligned} & \left[\begin{array}{cc} T^{-1} & 0 \\ 0 & T^T \end{array}\right] \left[\begin{array}{cc} A + BK & -BR^{-1}B^T \\ 0 & -(A + BK)^T \end{array}\right] \left[\begin{array}{cc} T & 0 \\ 0 & T^{-T} \end{array}\right] \\ = & \left[\begin{array}{cc} \Lambda & -T^{-1}BR^{-1}B^T T^{-T} \\ 0 & -\Lambda \end{array}\right] \end{aligned}$$

putting it together we get

$$\begin{aligned}
&\left[\begin{array}{cc} T^{-1} & 0 \\ 0 & T^T \end{array}\right]\left[\begin{array}{cc} I & 0 \\ -P & I \end{array}\right] H \left[\begin{array}{cc} I & 0 \\ P & I \end{array}\right]\left[\begin{array}{cc} T & 0 \\ 0 & T^{-T} \end{array}\right] \\
&= \left[\begin{array}{cc} T^{-1} & 0 \\ -T^T P & T^T \end{array}\right] H \left[\begin{array}{cc} T & 0 \\ PT & T^{-T} \end{array}\right] \\
&= \left[\begin{array}{cc} \Lambda & -T^{-1}BR^{-1}B^T T^{-T} \\ 0 & -\Lambda \end{array}\right]
\end{aligned}$$

and so

$$H \left[\begin{array}{c} T \\ PT \end{array}\right] = \left[\begin{array}{c} T \\ PT \end{array}\right] \Lambda$$

thus, the n columns of $\left[\begin{array}{c} T \\ PT \end{array}\right]$ are the eigenvectors of H associated with the stable eigenvalues $\lambda_1, \ldots, \lambda_n$

Solving ARE via Hamiltonian

- find eigenvalues of H, and let $\lambda_1, \ldots, \lambda_n$ denote the n stable ones (there are exactly n stable and n unstable ones)

- find associated eigenvectors $v_1, \ldots, v_n$, and partition as

$$\left[\begin{array}{ccc} v_1 & \cdots & v_n \end{array} \right] = \left[\begin{array}{c} X \\ Y \end{array} \right] \in \mathbf{R}^{2n \times n}$$

- $P = YX^{-1}$ is unique PSD solution of the ARE

(this is very close to the method used in practice, which does not require $A + BK$ to be diagonalizable)

Lecture 5
Observability and state estimation

- state estimation
- discrete-time observability
- observability – controllability duality
- observers for noiseless case
- continuous-time observability
- least-squares observers
- statistical interpretation
- example

State estimation set up

we consider the discrete-time system

$$x(t+1) = Ax(t) + Bu(t) + w(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

- w is state *disturbance* or *noise*
- v is sensor *noise* or *error*
- A, B, C, and D are known
- u and y are observed over time interval $[0, t-1]$
- w and v are not known, but can be described statistically or assumed small

State estimation problem

state estimation problem: estimate $x(s)$ from

$$u(0), \ldots, u(t-1), \; y(0), \ldots, y(t-1)$$

- $s = 0$: estimate initial state
- $s = t-1$: estimate current state
- $s = t$: estimate (*i.e.*, predict) next state

an algorithm or system that yields an estimate $\hat{x}(s)$ is called an *observer* or *state estimator*

$\hat{x}(s)$ is denoted $\hat{x}(s|t-1)$ to show what information estimate is based on (read, "$\hat{x}(s)$ given $t-1$")

Noiseless case

let's look at finding $x(0)$, with no state or measurement noise:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^p$

then we have

$$\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} = \mathcal{O}_t x(0) + \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix}$$

where

$$\mathcal{O}_t = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{t-1} \end{bmatrix}, \quad \mathcal{T}_t = \begin{bmatrix} D & 0 & \cdots & & \\ CB & D & 0 & \cdots & \\ \vdots & & & & \\ CA^{t-2}B & CA^{t-3}B & \cdots & CB & D \end{bmatrix}$$

- $\mathcal{O}_t$ maps initials state into resulting output over $[0, t-1]$
- $\mathcal{T}_t$ maps input to output over $[0, t-1]$

hence we have

$$\mathcal{O}_t x(0) = \begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix}$$

RHS is known, $x(0)$ is to be determined hence:

- can uniquely determine $x(0)$ if and only if $\mathcal{N}(\mathcal{O}_t) = \{0\}$

- $\mathcal{N}(\mathcal{O}_t)$ gives ambiguity in determining $x(0)$

- if $x(0) \in \mathcal{N}(\mathcal{O}_t)$ and $u = 0$, output is zero over interval $[0, t-1]$

- input u does not affect ability to determine $x(0)$;
  its effect can be subtracted out

Observability matrix

by C-H theorem, each A^k is linear combination of $A^0, \ldots, A^{n-1}$

hence for $t \geq n$, $\mathcal{N}(\mathcal{O}_t) = \mathcal{N}(\mathcal{O})$ where

$$\mathcal{O} = \mathcal{O}_n = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is called the *observability matrix*

if $x(0)$ can be deduced from u and y over $[0, t-1]$ for any t, then $x(0)$ can be deduced from u and y over $[0, n-1]$

$\mathcal{N}(\mathcal{O})$ is called *unobservable subspace*; describes ambiguity in determining state from input and output

system is called *observable* if $\mathcal{N}(\mathcal{O}) = \{0\}$, $i.e.$, $\mathbf{Rank}(\mathcal{O}) = n$

Observability – controllability duality

let $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ be dual of system (A, B, C, D), $i.e.$,

$$\tilde{A} = A^T, \quad \tilde{B} = C^T, \quad \tilde{C} = B^T, \quad \tilde{D} = D^T$$

controllability matrix of dual system is

$$\begin{aligned} \tilde{\mathcal{C}} &= [\tilde{B} \ \tilde{A}\tilde{B} \cdots \tilde{A}^{n-1}\tilde{B}] \\ &= [C^T \ A^T C^T \cdots (A^T)^{n-1} C^T] \\ &= \mathcal{O}^T, \end{aligned}$$

transpose of observability matrix

similarly we have $\tilde{\mathcal{O}} = \mathcal{C}^T$

thus, system is observable (controllable) if and only if dual system is controllable (observable)

in fact,

$$\mathcal{N}(\mathcal{O}) = \mathrm{range}(\mathcal{O}^T)^\perp = \mathrm{range}(\tilde{\mathcal{C}})^\perp$$

$i.e.$, unobservable subspace is orthogonal complement of controllable subspace of dual

Observers for noiseless case

suppose $\mathbf{Rank}(\mathcal{O}_t) = n$ ($i.e.$, system is observable) and let F be any left inverse of $\mathcal{O}_t$, $i.e.$, $F\mathcal{O}_t = I$

then we have the observer

$$x(0) = F\left(\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix}\right)$$

which deduces $x(0)$ (exactly) from u, y over $[0, t-1]$

in fact we have

$$x(\tau - t + 1) = F\left(\begin{bmatrix} y(\tau - t + 1) \\ \vdots \\ y(\tau) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(\tau - t + 1) \\ \vdots \\ u(\tau) \end{bmatrix}\right)$$

$i.e.$, our observer estimates what state was $t-1$ epochs ago, given past $t-1$ inputs & outputs

observer is (multi-input, multi-output) *finite impulse response* (FIR) filter, with inputs u and y, and output $\hat{x}$

Invariance of unobservable set

fact: the unobservable subspace $\mathcal{N}(\mathcal{O})$ is invariant, *i.e.*, if $z \in \mathcal{N}(\mathcal{O})$, then $Az \in \mathcal{N}(\mathcal{O})$

proof: suppose $z \in \mathcal{N}(\mathcal{O})$, *i.e.*, $CA^k z = 0$ for $k = 0, \ldots, n-1$

evidently $CA^k(Az) = 0$ for $k = 0, \ldots, n-2$;

$$CA^{n-1}(Az) = CA^n z = -\sum_{i=0}^{n-1} \alpha_i CA^i z = 0$$

(by C-H) where

$$\det(sI - A) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0$$

Continuous-time observability

continuous-time system with no sensor or state noise:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

can we deduce state x from u and y?

let's look at derivatives of y:

$$\begin{aligned} y &= Cx + Du \\ \dot{y} &= C\dot{x} + D\dot{u} = CAx + CBu + D\dot{u} \\ \ddot{y} &= CA^2x + CABu + CB\dot{u} + D\ddot{u} \end{aligned}$$

and so on

hence we have

$$\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} = \mathcal{O}x + \mathcal{T} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

where $\mathcal{O}$ is the observability matrix and

$$\mathcal{T} = \begin{bmatrix} D & 0 & \cdots & & \\ CB & D & 0 & \cdots & \\ \vdots & & & & \\ CA^{n-2}B & CA^{n-3}B & \cdots & CB & D \end{bmatrix}$$

(same matrices we encountered in discrete-time case!)

rewrite as

$$\mathcal{O}x = \begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \mathcal{T} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

RHS is known; x is to be determined

hence if $\mathcal{N}(\mathcal{O}) = \{0\}$ we can deduce $x(t)$ from derivatives of $u(t)$, $y(t)$ up to order $n-1$

in this case we say system is observable

can construct an observer using any left inverse F of $\mathcal{O}$:

$$x = F\left(\begin{bmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{bmatrix} - \mathcal{T} \begin{bmatrix} u \\ \dot{u} \\ \vdots \\ u^{(n-1)} \end{bmatrix} \right)$$

- reconstructs $x(t)$ (exactly and instantaneously) from

$$u(t), \ldots, u^{(n-1)}(t), \; y(t), \ldots, y^{(n-1)}(t)$$

- derivative-based state reconstruction is dual of state transfer using impulsive inputs

A converse

suppose $z \in \mathcal{N}(\mathcal{O})$ (the unobservable subspace), and u is any input, with x, y the corresponding state and output, *i.e.*,

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

then state trajectory $\tilde{x} = x + e^{At}z$ satisfies

$$\dot{\tilde{x}} = A\tilde{x} + Bu, \quad y = C\tilde{x} + Du$$

i.e., input/output signals u, y consistent with both state trajectories x, $\tilde{x}$

hence if system is unobservable, no signal processing of any kind applied to u and y can deduce x

unobservable subspace $\mathcal{N}(\mathcal{O})$ gives fundamental ambiguity in deducing x from u, y

Least-squares observers

discrete-time system, with sensor noise:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

we assume $\mathbf{Rank}(\mathcal{O}_t) = n$ (hence, system is observable)

least-squares observer uses pseudo-inverse:

$$\hat{x}(0) = \mathcal{O}_t^\dagger \left(\begin{bmatrix} y(0) \\ \vdots \\ y(t-1) \end{bmatrix} - \mathcal{T}_t \begin{bmatrix} u(0) \\ \vdots \\ u(t-1) \end{bmatrix} \right)$$

where $\mathcal{O}_t^\dagger = \left(\mathcal{O}_t^T \mathcal{O}_t\right)^{-1} \mathcal{O}_t^T$

since $\mathcal{O}_t^\dagger \mathcal{O}_t = I$, we have

$$\hat{x}_{\rm ls}(0) = x(0) + \mathcal{O}_t^\dagger \left[\begin{array}{c} v(0) \\ \vdots \\ v(t-1) \end{array} \right]$$

in particular, $\hat{x}_{\rm ls}(0) = x(0)$ if sensor noise is zero
($i.e.$, observer recovers exact state in noiseless case)

interpretation: $\hat{x}_{\rm ls}(0)$ minimizes discrepancy between

- output $\hat{y}$ that *would be* observed, with input u and initial state $x(0)$ (and no sensor noise), and
- output y that *was* observed,

measured as $\displaystyle\sum_{\tau=0}^{t-1} \|\hat{y}(\tau) - y(\tau)\|^2$

can express least-squares initial state estimate as

$$\hat{x}_{\rm ls}(0) = \left(\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau\right)^{-1} \sum_{\tau=0}^{t-1} (A^T)^\tau C^T \tilde{y}(\tau)$$

where $\tilde{y}$ is observed output with portion due to input subtracted: $\tilde{y} = y - h * u$ where h is impulse response

Statistical interpretation of least-squares observer

suppose sensor noise is IID $\mathcal{N}(0, \sigma I)$

- called *white noise*
- each sensor has noise variance σ

then $\hat{x}_{\rm ls}(0)$ is MMSE estimate of $x(0)$ when $x(0)$ is deterministic (or has 'infinite' prior variance)

estimation error $z = \hat{x}_{\rm ls}(0) - x(0)$ can be expressed as

$$z = \mathcal{O}_t^\dagger \begin{bmatrix} v(0) \\ \vdots \\ v(t-1) \end{bmatrix}$$

hence $z \ \sim \ \mathcal{N}\left(0, \sigma \mathcal{O}^\dagger \mathcal{O}^{\dagger T}\right)$

$i.e.$, covariance of least-squares initial state estimation error is

$$\sigma \mathcal{O}^{\dagger} \mathcal{O}^{\dagger T} = \sigma \left(\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \right)^{-1}$$

we'll assume $\sigma = 1$ to simplify

matrix $\left(\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \right)^{-1}$ gives measure of 'how observable' the state is, over $[0, t-1]$

Infinite horizon error covariance

the matrix

$$P = \lim_{t\to\infty} \left(\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau \right)^{-1}$$

always exists, and gives the limiting error covariance in estimating $x(0)$ from u, y over longer and longer periods:

$$\lim_{t\to\infty} \ \mathbf{E}(\hat{x}_{\rm ls}(0|t-1) - x(0))(\hat{x}_{\rm ls}(0|t-1) - x(0))^T = P$$

- if A is stable, $P > 0$
  $i.e.$, can't estimate initial state perfectly even with infinite number of measurements $u(t),\ y(t)$, $t = 0, \ldots$ (since memory of $x(0)$ fades . . .)
- if A is not stable, then P can have nonzero nullspace
  $i.e.$, initial state estimation error gets arbitrarily small (at least in some directions) as more and more of signals u and y are observed

Observability Gramian

suppose system

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is observable and stable

then $\displaystyle\sum_{\tau=0}^{t-1} (A^T)^\tau C^T C A^\tau$ converges as $t \to \infty$ since A^τ decays geometrically

the matrix $W_o = \displaystyle\sum_{\tau=0}^{\infty} (A^T)^\tau C^T C A^\tau$ is called the *observability Gramian*

W_o satisfies the matrix equation

$$W_o - A^T W_o A = C^T C$$

which is called the observability *Lyapunov equation* (and can be solved exactly and efficiently)

Current state estimation

we have concentrated on estimating $x(0)$ from

$$u(0), \ldots, u(t-1),\ y(0), \ldots, y(t-1)$$

now we look at estimating $x(t-1)$ from this data

we assume

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t) + v(t)$$

- no state noise
- v is white, *i.e.*, IID $\mathcal{N}(0, \sigma I)$

using

$$x(t-1) = A^{t-1}x(0) + \sum_{\tau=0}^{t-2} A^{t-2-\tau} Bu(\tau)$$

we get current state least-squares estimator:

$$\hat{x}(t-1|t-1) = A^{t-1}\hat{x}_{\rm ls}(0|t-1) + \sum_{\tau=0}^{t-2} A^{t-2-\tau}Bu(\tau)$$

righthand term ($i.e.$, effect of input on current state) is known

estimation error $z = \hat{x}(t-1|t-1) - x(t-1)$ can be expressed as

$$z = A^{t-1}\mathcal{O}_t^\dagger \left[\begin{array}{c} v(0) \\ \vdots \\ v(t-1) \end{array}\right]$$

hence $z \;\sim\; \mathcal{N}\left(0, \sigma A^{t-1}\mathcal{O}^\dagger\mathcal{O}^{\dagger T}(A^T)^{t-1}\right)$

$i.e.$, covariance of least-squares current state estimation error is

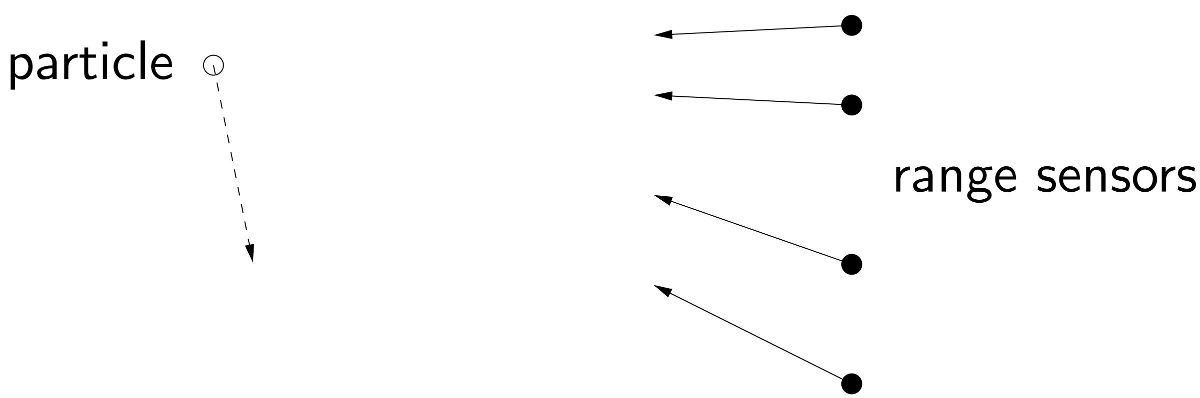
$$\sigma A^{t-1}\mathcal{O}^\dagger\mathcal{O}^{\dagger T}(A^T)^{t-1} = \sigma A^{t-1}\left(\sum_{\tau=0}^{t-1}(A^T)^\tau C^T C A^\tau\right)^{-1}(A^T)^{t-1}$$

this matrix measures 'how observable' current state is, from past t inputs & outputs

- decreases (in matrix sense) as t increases
- hence has limit as $t \to \infty$ (gives limiting error covariance of estimating current state given all past inputs & outputs)

Example

- particle in $\mathbf{R}^2$ moves with uniform velocity
- (linear, noisy) range measurements from directions $-15°$, $0°$, $20°$, $30°$, once per second
- range noises IID $\mathcal{N}(0,1)$
- no assumptions about initial position & velocity



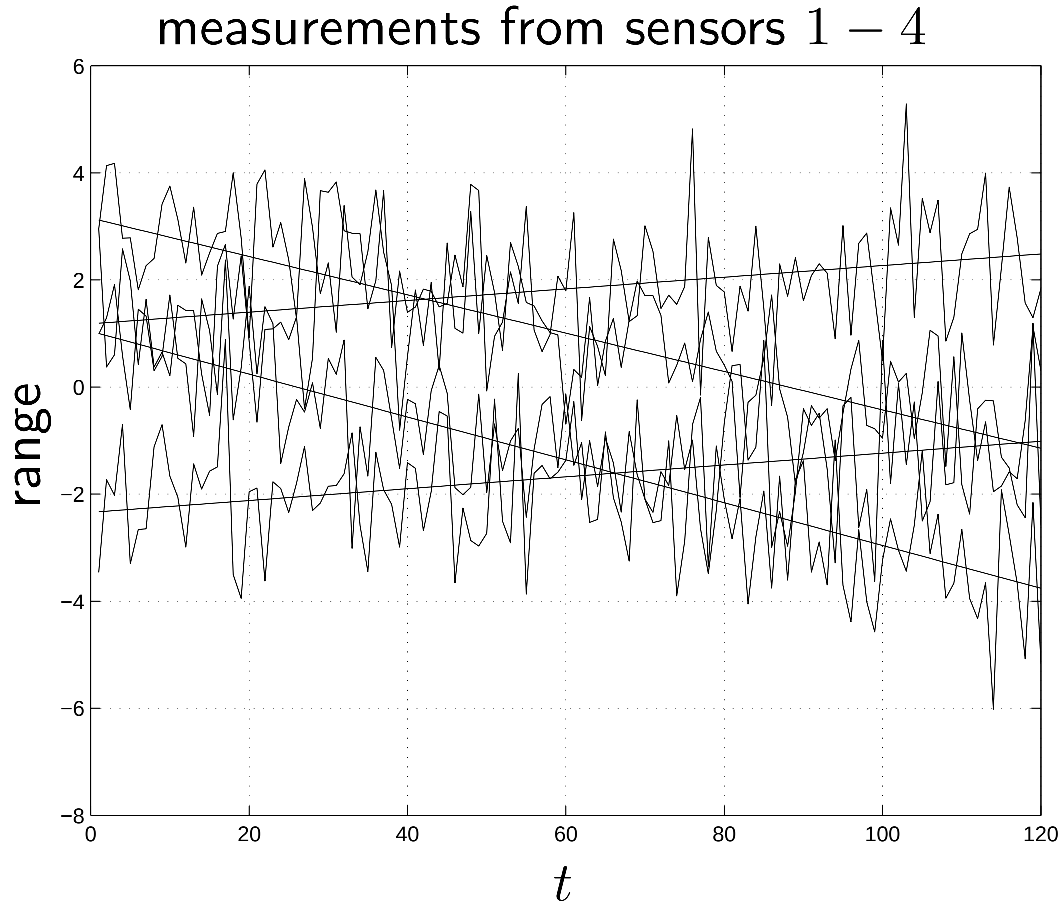
problem: estimate initial position & velocity from range measurements express as linear system

$$x(t+1) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t), \qquad y(t) = \begin{bmatrix} k_1^T \\ \vdots \\ k_4^T \end{bmatrix} x(t) + v(t)$$

- $(x_1(t), x_2(t))$ is position of particle
- $(x_3(t), x_4(t))$ is velocity of particle
- $v(t) \sim \mathcal{N}(0, I)$
- k_i is unit vector from sensor i to origin

true initial position & velocities: $x(0) = (1 \ -3 \ -0.04 \ 0.03)$

range measurements (& noiseless versions):



- estimate based on $(y(0), \ldots, y(t))$ is $\hat{x}(0|t)$

- actual RMS position error is

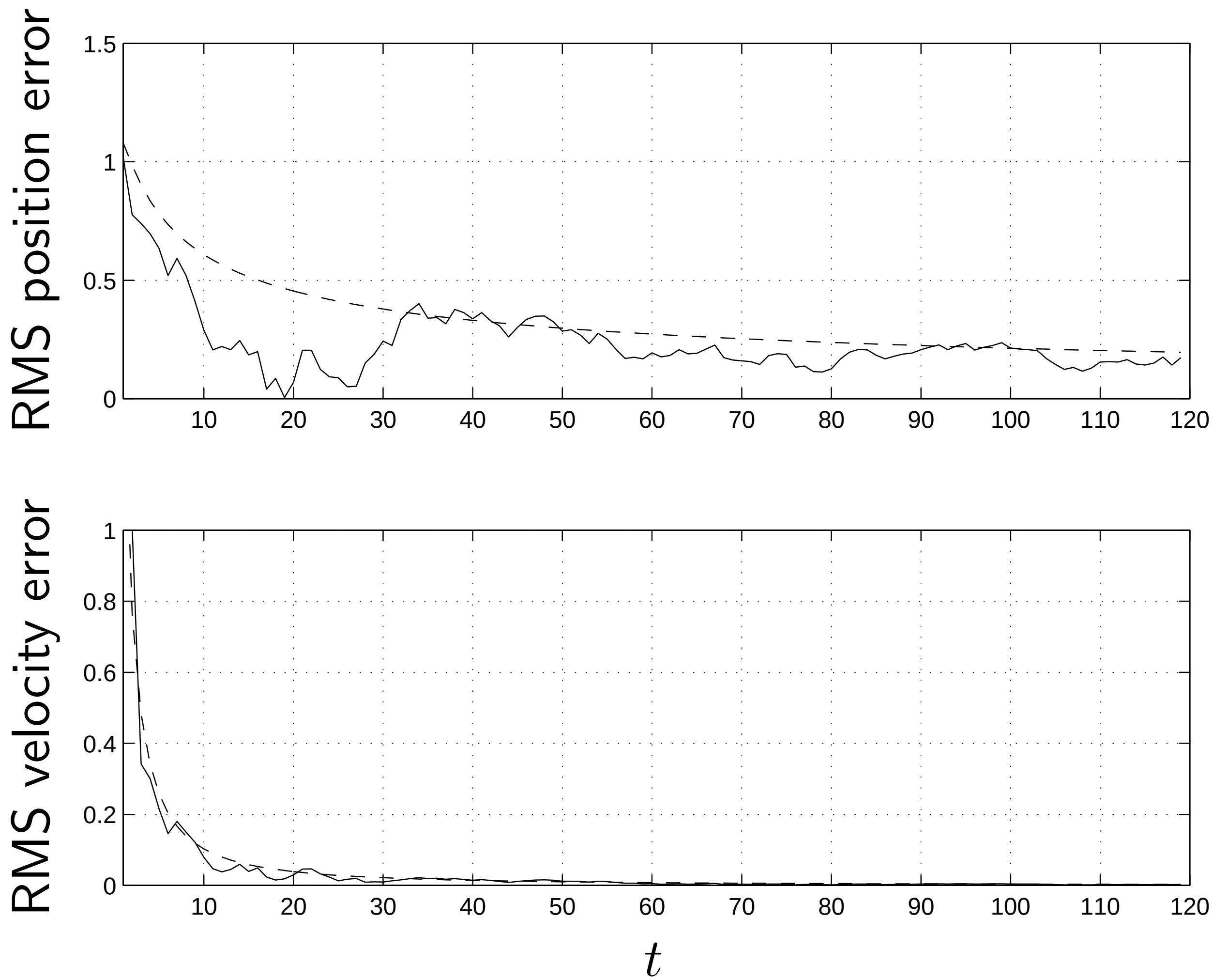
$$\sqrt{(\hat{x}_1(0|t) - x_1(0))^2 + (\hat{x}_2(0|t) - x_2(0))^2}$$

(similarly for actual RMS velocity error)

- position error std. deviation is

$$\sqrt{\mathbf{E}\left((\hat{x}_1(0|t) - x_1(0))^2 + (\hat{x}_2(0|t) - x_2(0))^2\right)}$$

(similarly for velocity)


RMS position error
1.5
1
0.5
0
10 20 30 40 50 60 70 80 90 100 110 120
RMS velocity error
1
0.8
0.6
0.4
0.2
0
10 20 30 40 50 60 70 80 90 100 110 120
t

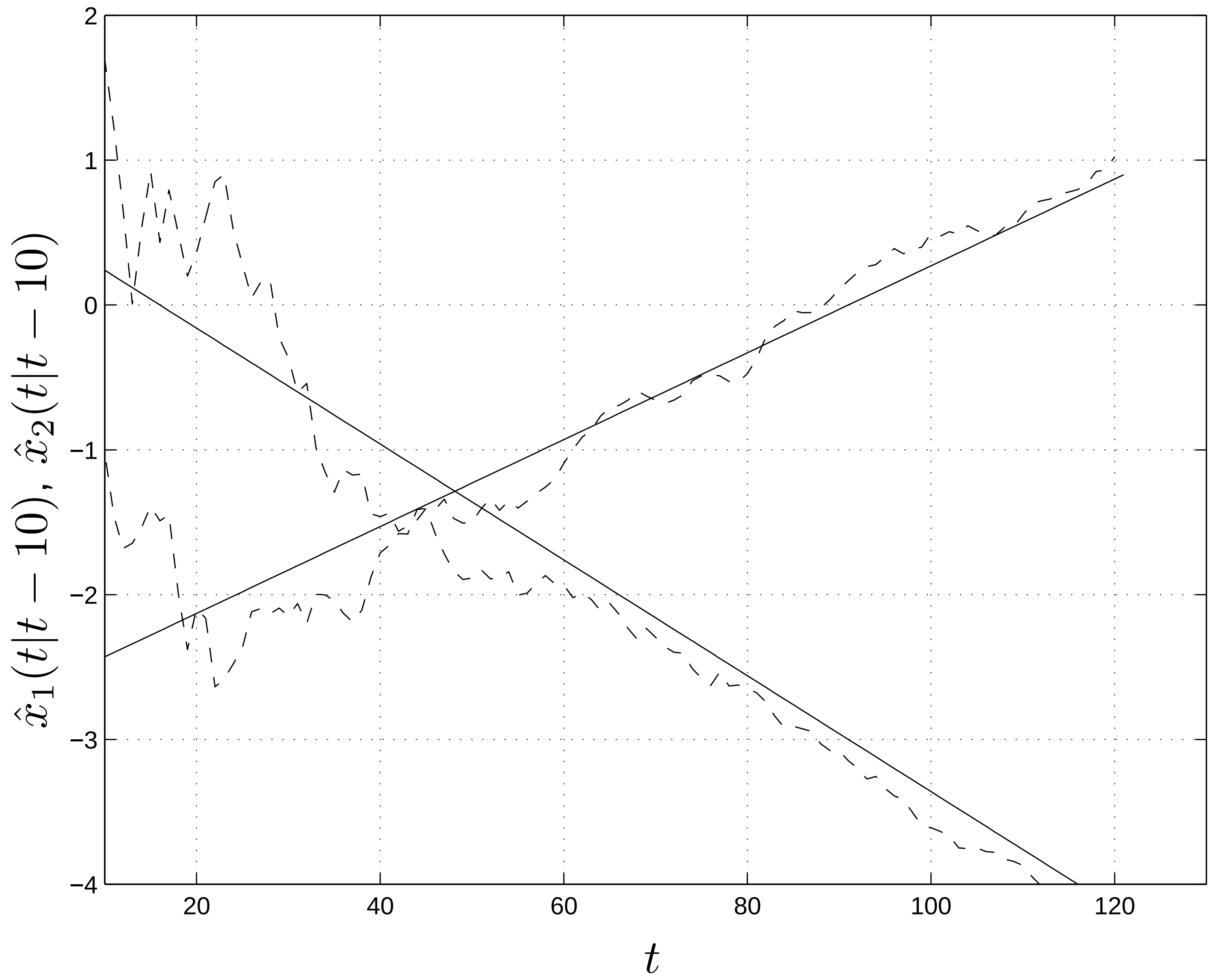
Example ctd: state prediction

predict particle position 10 seconds in future:

$$\hat{x}(t+10|t) = A^{t+10}\hat{x}_{\text{ls}}(0|t)$$

$$x(t+10) = A^{t+10}x(0)$$

plot shows estimates (dashed), and actual value (solid) of position of particle 10 steps ahead, for $10 \leq t \leq 110$



2
1
0
−1
−2
−3
−4
20
40
60
80
100
120
$\hat{x}_1(t|t-10)$, $\hat{x}_2(t|t-10)$
t

Continuous-time least-squares state estimation

assume $\dot{x} = Ax + Bu$, $y = Cx + Du + v$ is observable

least-squares observer is

$$\hat{x}_{\rm ls}(0) = \left(\int_0^t e^{A^T\tau} C^T C e^{A\tau} \, d\tau \right)^{-1} \int_0^t e^{A^T\bar{t}} C^T \tilde{y}(\bar{t}) \, d\bar{t}$$

where $\tilde{y} = y - h * u$ is observed output minus part due to input

then $\hat{x}_{\rm ls}(0) = x(0)$ if $v = 0$

$\hat{x}_{\rm ls}(0)$ is limiting MMSE estimate when $v(t) \sim \mathcal{N}(0, \sigma I)$ and $\mathbf{E}\, v(t)v(s)^T = 0$ unless $t - s$ is very small

(called white noise — a tricky concept)

Lecture 5
Invariant subspaces

- invariant subspaces
- a matrix criterion
- Sylvester equation
- the PBH controllability and observability conditions
- invariant subspaces, quadratic matrix equations, and the ARE

Invariant subspaces

suppose $A \in \mathbf{R}^{n\times n}$ and $\mathcal{V} \subseteq \mathbf{R}^n$ is a subspace

we say that $\mathcal{V}$ is A-*invariant* if $A\mathcal{V} \subseteq \mathcal{V}$, $i.e.$, $v \in \mathcal{V} \implies Av \in \mathcal{V}$

examples:

- $\{0\}$ and $\mathbf{R}^n$ are always A-invariant
- $\operatorname{span}\{v_1, \ldots, v_m\}$ is A-invariant, where v_i are (right) eigenvectors of A
- if A is block upper triangular,

$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ 0 & A_{22} \end{array}\right],$$

with $A_{11} \in \mathbf{R}^{r\times r}$, then $\mathcal{V} = \left\{ \left[\begin{array}{c} z \\ 0 \end{array}\right] \,\middle|\, z \in \mathbf{R}^r \right\}$ is A-invariant

Examples from linear systems

- if $B \in \mathbf{R}^{n\times m}$, then the controllable subspace

$$\mathcal{R}(\mathcal{C}) = \mathcal{R}\left([B\ AB\ \cdots\ A^{n-1}B]\right)$$

  is A-invariant

- if $C \in \mathbf{R}^{p\times n}$, then the unobservable subspace

$$\mathcal{N}(\mathcal{O}) = \mathcal{N}\left(\left[\begin{array}{c} C \\ \vdots \\ CA^{n-1} \end{array}\right]\right)$$

  is A-invariant

Dynamical interpretation

consider system $\dot{x} = Ax$

$\mathcal{V}$ is A-invariant if and only if

$$x(0) \in \mathcal{V} \implies x(t) \in \mathcal{V} \text{ for all } t \geq 0$$

(same statement holds for discrete-time system)

A matrix criterion for A-invariance

suppose $\mathcal{V}$ is A-invariant

let columns of $M \in \mathbf{R}^{n \times k}$ span $\mathcal{V}$, *i.e.*,

$$\mathcal{V} = \mathcal{R}(M) = \mathcal{R}([t_1 \ \cdots \ t_k])$$

since $At_1 \in \mathcal{V}$, we can express it as

$$At_1 = x_{11}t_1 + \cdots + x_{k1}t_k$$

we can do the same for $At_2, \ldots, At_k$, which gives

$$A[t_1 \ \cdots \ t_k] = [t_1 \ \cdots \ t_k] \begin{bmatrix} x_{11} & \cdots & x_{1k} \\ \vdots & & \vdots \\ x_{k1} & \cdots & x_{kk} \end{bmatrix}$$

or, simply, $AM = MX$

in other words: if $\mathcal{R}(M)$ is A-invariant, then there is a matrix X such that $AM = MX$

converse is also true: if there is an X such that $AM = MX$, then $\mathcal{R}(M)$ is A-invariant

now assume M is rank k, *i.e.*, $\{t_1, \ldots, t_k\}$ is a basis for $\mathcal{V}$

then every eigenvalue of X is an eigenvalue of A, and the associated eigenvector is in $\mathcal{V} = \mathcal{R}(M)$

if $Xu = \lambda u$, $u \neq 0$, then $Mu \neq 0$ and $A(Mu) = MXu = \lambda Mu$

so the eigenvalues of X are a subset of the eigenvalues of A

more generally: if $AM = MX$ (no assumption on rank of M), then A and X share at least $\mathbf{Rank}(M)$ eigenvalues

Sylvester equation

the *Sylvester equation* is $AX + XB = C$, where $A,\ B,\ C,\ X \in \mathbf{R}^{n\times n}$

when does this have a solution X for every C?

express as $S(X) = C$, where S is the linear function $S(X) = AX + XB$ (S maps $\mathbf{R}^{n\times n}$ into $\mathbf{R}^{n\times n}$ and is called the *Sylvester operator*)

so the question is: when is S nonsingular?

S is singular if and only if there exists a nonzero X with $S(X) = 0$

this means $AX + XB = 0$, so $AX = X(-B)$, which means A and $-B$ share at least one eigenvalue (since $X \neq 0$)

so we have: if S is singular, then A and $-B$ have a common eigenvalue let's show the converse: if A and $-B$ share an eigenvalue, S is singular

suppose

$$Av = \lambda v, \qquad w^T B = -\lambda w^T, \qquad v,\ w \neq 0$$

then with $X = vw^T$ we have $X \neq 0$ and

$$S(X) = AX + XB = Avw^T + vw^T B = (\lambda v)w^T + v(-\lambda w^T) = 0$$

which shows S is singular

so, Sylvestor operator is singular if and only if A and $-B$ have a common eigenvalue

or: Sylvestor operator is nonsingular if and only if A and $-B$ have no common eigenvalues

Uniqueness of stabilizing ARE solution

suppose P is any solution of ARE

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

and define $K = -R^{-1}B^T P$

we say P is a *stabilizing solution* of ARE if

$$A + BK = A - BR^{-1}B^T P$$

is stable, $i.e.$, its eigenvalues have negative real part

fact: there is at most one stabilizing solution of the ARE
(which therefore is the one that gives the value function)

to show this, suppose P_1 and P_2 are both stabilizing solutions

subtract AREs to get

$$A^T(P_1 - P_2) + (P_1 - P_2)A - P_1BR^{-1}B^TP_1 + P_2BR^{-1}B^TP_2 = 0$$

rewrite as Sylvester equation

$$(A + BK_2)^T(P_1 - P_2) + (P_1 - P_2)(A + BK_1) = 0$$

since $A + BK_2$ and $A + BK_1$ are both stable, $A + BK_2$ and $-(A + BK_1)$ cannot share any eigenvalues, so we conclude $P_1 - P_2 = 0$

Change of coordinates

suppose $\mathcal{V} = \mathcal{R}(M)$ is A-invariant, where $M \in \mathbf{R}^{n \times k}$ is rank k

find $\tilde{M} \in \mathbf{R}^{n \times (n-k)}$ so that $[M \ \tilde{M}]$ is nonsingular

$$A[M \ \tilde{M}] = [AM \ A\tilde{M}] = [M \ \tilde{M}] \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

where

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = [M \ \tilde{M}]^{-1} A\tilde{M}$$

with $T = [M \ \tilde{M}]$, we have

$$T^{-1}AT = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$$

in other words: if $\mathcal{V}$ is A-invariant we can change coordinates so that

- A becomes block upper triangular in the new coordinates
- $\mathcal{V}$ corresponds to $\left\{ \begin{bmatrix} z \\ 0 \end{bmatrix} \;\middle|\; z \in \mathbf{R}^k \right\}$ in the new coordinates

Revealing the controllable subspace

consider $\dot{x} = Ax + Bu$ (or $x(t+1) = Ax(t) + Bu(t)$) and assume it is *not* controllable, so $\mathcal{V} = \mathcal{R}(\mathcal{C}) \neq \mathbf{R}^n$

let columns of $M \in \mathbf{R}^k$ be basis for controllable subspace ($e.g.$, choose k independent columns from $\mathcal{C}$)

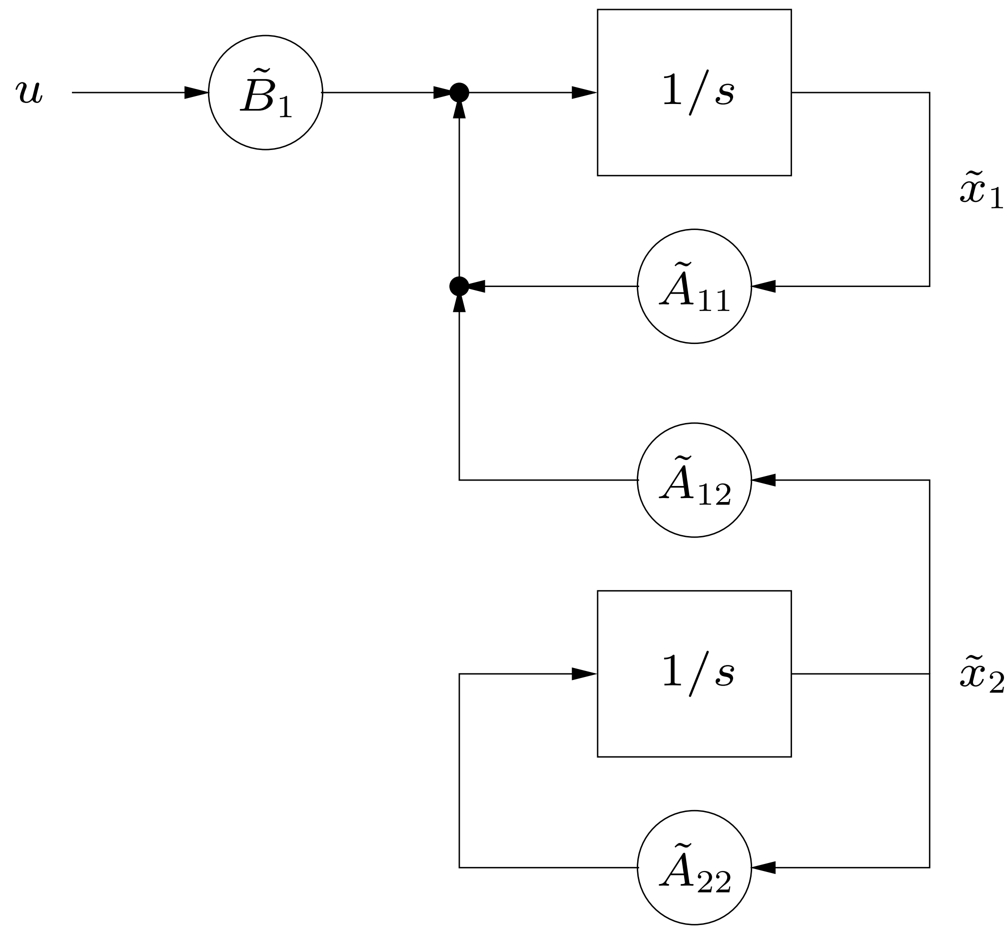
let $\tilde{M} \in \mathbf{R}^{n \times (n-k)}$ be such that $T = [M \ \tilde{M}]$ is nonsingular

then

$$T^{-1}AT = \left[\begin{array}{cc} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{array}\right], \qquad T^{-1}B = \left[\begin{array}{c} \tilde{B}_1 \\ 0 \end{array}\right]$$

$$\tilde{\mathcal{C}} = T^{-1}\mathcal{C} = \left[\begin{array}{ccc} \tilde{B}_1 & \cdots & \tilde{A}_{11}^{n-1}\tilde{B}_1 \\ 0 & \cdots & 0 \end{array}\right]$$

in the new coordinates the controllable subspace is $\{(z, 0) \mid z \in \mathbf{R}^k\}$; $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable we have changed coordinates to reveal the controllable subspace:



roughly speaking, $\tilde{x}_1$ is the controllable part of the state

Revealing the unobservable subspace

similarly, if (C, A) is not observable, we can change coordinates to obtain

$$T^{-1}AT = \left[\begin{array}{cc} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{array} \right], \qquad CT = \left[\begin{array}{cc} \tilde{C}_1 & 0 \end{array} \right]$$

and $(\tilde{C}_1, \tilde{A}_{11})$ is observable

Popov-Belevitch-Hautus controllability test

PBH controllability criterion: (A, B) is controllable if and only if

$$\mathbf{Rank}\ [sI - A\ B] = n \text{ for all } s \in \mathbf{C}$$

equivalent to:

(A, B) is uncontrollable if and only if there is a $w \neq 0$ with

$$w^T A = \lambda w^T, \qquad w^T B = 0$$

$i.e.$, a left eigenvector is orthogonal to columns of B

to show it, first assume that $w \neq 0$, $w^T A = \lambda w^T$, $w^T B = 0$

then for $k = 1, \ldots, n-1$, $w^T A^k B = \lambda^k w^T B = 0$, so

$$w^T [B \ AB \ \cdots \ A^{n-1}B] = w^T \mathcal{C} = 0$$

which shows (A, B) not controllable

conversely, suppose (A, B) not controllable

change coordinates as on p.5–15, let z be any left eigenvector of $\tilde{A}_{22}$, and define $\tilde{w} = (0, z)$

then $\tilde{w}^T \tilde{A} = \lambda \tilde{w}^T$, $\tilde{w}^T \tilde{B} = 0$

it follows that $w^T A = \lambda w^T$, $w^T B = 0$, where $w = T^{-T} \tilde{w}$

PBH observability test

PBH observability criterion: (C, A) is observable if and only if

$$\mathbf{Rank} \begin{bmatrix} sI - A \\ C \end{bmatrix} = n \text{ for all } s \in \mathbf{C}$$

equivalent to:

(C, A) is unobservable if and only if there is a $v \neq 0$ with

$$Av = \lambda v, \qquad Cv = 0$$

i.e., a (right) eigenvector is in the nullspace of C

Observability and controllability of modes

the PBH tests allow us to identify unobservable and uncontrollable modes

the mode associated with right and left eigenvectors $v,\ w$ is

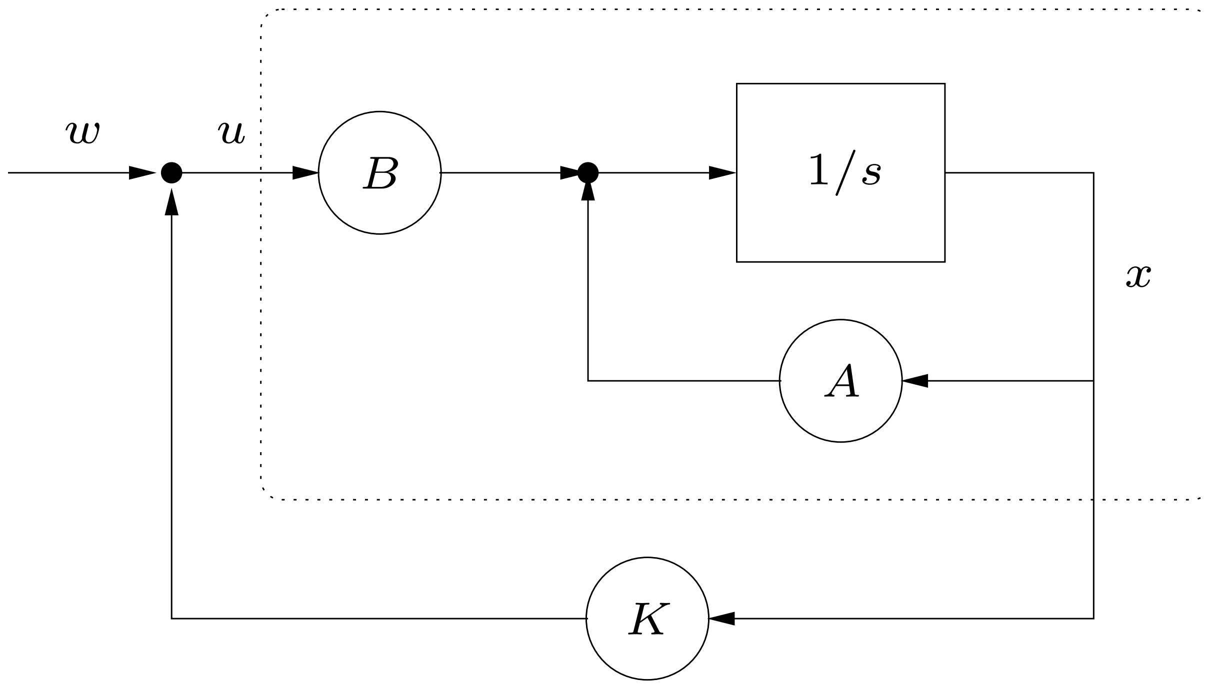
- uncontrollable if $w^T B = 0$
- unobservable if $Cv = 0$

(classification can be done with repeated eigenvalues, Jordan blocks, but gets tricky)

Controllability and linear state feedback

we consider system $\dot{x} = Ax + Bu$ (or $x(t+1) = Ax(t) + Bu(t)$)

we refer to $u = Kx + w$ as a *linear state feedback* (with auxiliary input w), with associated *closed-loop system* $\dot{x} = (A + BK)x + Bw$



suppose $w^T A = \lambda w^T$, $w \neq 0$, $w^T B = 0$, $i.e.$, w corresponds to uncontrollable mode of open loop system

then $w^T(A+BK) = w^T A + w^T BK = \lambda w^T$, $i.e.$, w is also a left eigenvector of closed-loop system, associated with eigenvalue λ

$i.e.$, eigenvalues (and indeed, left eigenvectors) associated with uncontrollable modes cannot be changed by linear state feedback

conversely, if w is left eigenvector associated with uncontrollable closed-loop mode, then w is left eigenvector associated with uncontrollable open-loop mode

in other words: state feedback preserves uncontrollable eigenvalues and the associated left eigenvectors

Invariant subspaces and quadratic matrix equations

suppose $\mathcal{V} = \mathcal{R}(M)$ is A-invariant, where $M \in \mathbf{R}^{n\times k}$ is rank k, so $AM = MX$ for some $X \in \mathbf{R}^{k\times k}$

conformably partition as

$$\left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right] \left[\begin{array}{c} M_1 \\ M_2 \end{array}\right] = \left[\begin{array}{c} M_1 \\ M_2 \end{array}\right] X$$

$$A_{11}M_1 + A_{12}M_2 = M_1X, \qquad A_{21}M_1 + A_{22}M_2 = M_2X$$

eliminate X from first equation (assuming M_1 is nonsingular):

$$X = M_1^{-1}A_{11}M_1 + M_1^{-1}A_{12}M_2$$

substituting this into second equation yields

$$A_{21}M_1 + A_{22}M_2 = M_2M_1^{-1}A_{11}M_1 + M_2M_1^{-1}A_{12}M_2$$

multiply on right by M_1^{-1}:

$$A_{21} + A_{22}M_2M_1^{-1} = M_2M_1^{-1}A_{11} + M_2M_1^{-1}A_{12}M_2M_1^{-1}$$

with $P = M_2M_1^{-1}$, we have

$$-A_{22}P + PA_{11} - A_{21} + PA_{12}P = 0,$$

a general quadratic matrix equation

if we take A to be Hamitonian associated with a cts-time LQR problem, we recover the method of solving ARE via stable eigenvectors of Hamiltonian

Lecture 6
Estimation

- Gaussian random vectors
- minimum mean-square estimation (MMSE)
- MMSE with linear measurements
- relation to least-squares, pseudo-inverse

Gaussian random vectors

random vector $x \in \mathbf{R}^n$ is *Gaussian* if it has density

$$p_x(v) = (2\pi)^{-n/2}(\det \Sigma)^{-1/2} \exp\left(-\frac{1}{2}(v-\bar{x})^T\Sigma^{-1}(v-\bar{x})\right),$$

for some $\Sigma = \Sigma^T > 0$, $\bar{x} \in \mathbf{R}^n$

- denoted $x \sim \mathcal{N}(\bar{x}, \Sigma)$
- $\bar{x} \in \mathbf{R}^n$ is the *mean* or *expected* value of x, $i.e.$,

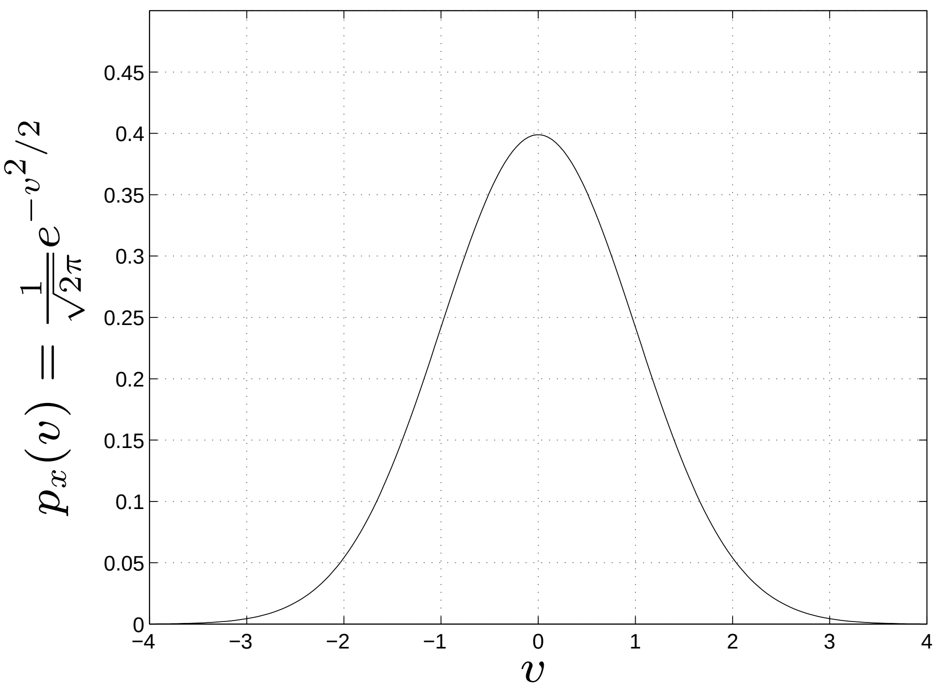
$$\bar{x} = \mathbf{E}\, x = \int v p_x(v) dv$$

- $\Sigma = \Sigma^T > 0$ is the *covariance* matrix of x, $i.e.$,

$$\Sigma \quad = \quad \mathbf{E}(x-\bar{x})(x-\bar{x})^T$$

$$
\begin{aligned}
&= \mathbf{E}\, xx^T - \bar{x}\bar{x}^T \\
&= \int (v-\bar{x})(v-\bar{x})^T p_x(v)\, dv
\end{aligned}
$$

density for $x \sim \mathcal{N}(0,1)$:



- mean and variance of scalar random variable x_i are

$$\mathbf{E}\, x_i = \bar{x}_i, \quad \mathbf{E}(x_i - \bar{x}_i)^2 = \Sigma_{ii}$$

  hence standard deviation of x_i is $\sqrt{\Sigma_{ii}}$

- covariance between x_i and x_j is $\mathbf{E}(x_i - \bar{x}_i)(x_j - \bar{x}_j) = \Sigma_{ij}$
- correlation coefficient between x_i and x_j is $\rho_{ij} = \dfrac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$
- mean (norm) square deviation of x from $\bar{x}$ is

$$\mathbf{E}\,\|x - \bar{x}\|^2 = \mathbf{E}\,\mathbf{Tr}(x - \bar{x})(x - \bar{x})^T = \mathbf{Tr}\,\Sigma = \sum_{i=1}^{n} \Sigma_{ii}$$

  (using $\mathbf{Tr}\, AB = \mathbf{Tr}\, BA$)

example: $x \sim \mathcal{N}(0, I)$ means x_i are independent identically distributed (IID) $\mathcal{N}(0, 1)$ random variables

Confidence ellipsoids

- $p_x(v)$ is constant for $(v-\bar{x})^T\Sigma^{-1}(v-\bar{x}) = \alpha$, *i.e.*, on the surface of ellipsoid
$$\mathcal{E}_\alpha = \{v \mid (v-\bar{x})^T\Sigma^{-1}(v-\bar{x}) \leq \alpha\}$$
  - thus $\bar{x}$ and Σ determine shape of density

- η*-confidence set* for random variable z is smallest volume set S with $\mathbf{Prob}(z \in S) \geq \eta$
  - in general case confidence set has form $\{v \; p_z(v) \geq \beta\}$

- $\mathcal{E}_\alpha$ are the η-confidence sets for Gaussian, called *confidence ellipsoids*
  - α determines confidence level η

Confidence levels

the nonnegative random variable $(x - \bar{x})^T \Sigma^{-1} (x - \bar{x})$ has a χ_n^2 distribution, so $\mathbf{Prob}(x \in \mathcal{E}_\alpha) = F_{\chi_n^2}(\alpha)$ where $F_{\chi_n^2}$ is the CDF

some good approximations:

- $\mathcal{E}_n$ gives about 50% probability
- $\mathcal{E}_{n+2\sqrt{n}}$ gives about 90% probability

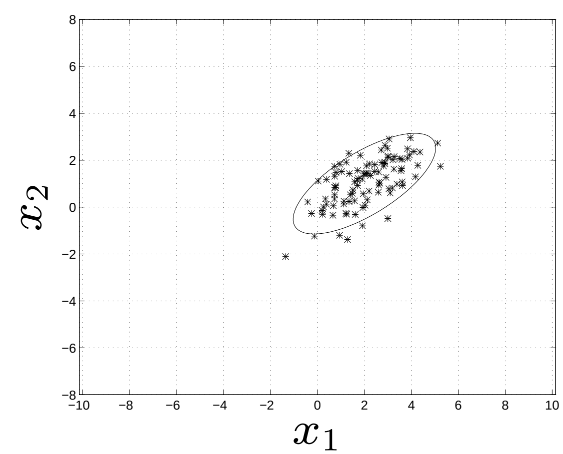
geometrically:

- mean $\bar{x}$ gives center of ellipsoid
- semiaxes are $\sqrt{\alpha\lambda_i}u_i$, where u_i are (orthonormal) eigenvectors of Σ with eigenvalues λ_i

example: $x \sim \mathcal{N}(\bar{x}, \Sigma)$ with $\bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

- x_1 has mean 2, std. dev. $\sqrt{2}$
- x_2 has mean 1, std. dev. 1
- correlation coefficient between x_1 and x_2 is $\rho = 1/\sqrt{2}$
- $\mathbf{E}\,\|x - \bar{x}\|^2 = 3$

90% confidence ellipsoid corresponds to $\alpha = 4.6$:



(here, 91 out of 100 fall in $\mathcal{E}_{4.6}$)

Affine transformation

suppose $x \sim \mathcal{N}(\bar{x}, \Sigma_x)$

consider affine transformation of x:

$$z = Ax + b,$$

where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$

then z is Gaussian, with mean

$$\mathbf{E}\, z = \mathbf{E}(Ax + b) = A\,\mathbf{E}\, x + b = A\bar{x} + b$$

and covariance

$$\begin{aligned} \Sigma_z &= \mathbf{E}(z - \bar{z})(z - \bar{z})^T \\ &= \mathbf{E}\, A(x - \bar{x})(x - \bar{x})^T A^T \\ &= A\Sigma_x A^T \end{aligned}$$

examples:

- if $w \sim \mathcal{N}(0, I)$ then $x = \Sigma^{1/2} w + \bar{x}$ is $\mathcal{N}(\bar{x}, \Sigma)$

  useful for simulating vectors with given mean and covariance

- conversely, if $x \sim \mathcal{N}(\bar{x}, \Sigma)$ then $z = \Sigma^{-1/2}(x - \bar{x})$ is $\mathcal{N}(0, I)$

  (normalizes & decorrelates; called *whitening* or *normalizing*) suppose $x \sim \mathcal{N}(\bar{x}, \Sigma)$ and $c \in \mathbf{R}^n$

scalar $c^T x$ has mean $c^T \bar{x}$ and variance $c^T \Sigma c$

thus (unit length) direction of minimum variability for x is u, where

$$\Sigma u = \lambda_{\min} u, \quad \|u\| = 1$$

standard deviation of $u_n^T x$ is $\sqrt{\lambda_{\min}}$

(similarly for maximum variability)

Degenerate Gaussian vectors

- it is convenient to allow Σ to be singular (but still $\Sigma = \Sigma^T \geq 0$)
  - in this case density formula obviously does not hold
  - meaning: in some directions x is not random at all
  - random variable x is called a *degenerate Gaussian*

- write Σ as

$$\Sigma = \left[\begin{array}{cc} Q_+ & Q_0 \end{array}\right] \left[\begin{array}{cc} \Sigma_+ & 0 \\ 0 & 0 \end{array}\right] \left[\begin{array}{cc} Q_+ & Q_0 \end{array}\right]^T$$

where $Q = [Q_+ \; Q_0]$ is orthogonal, $\Sigma_+ > 0$

- columns of Q_0 are orthonormal basis for $\mathcal{N}(\Sigma)$
- columns of Q_+ are orthonormal basis for $\mathrm{range}(\Sigma)$

- then

$$Q^T x = \begin{bmatrix} z \\ w \end{bmatrix}, \qquad x = Q_+ z + Q_0 w$$

  - $z \sim \mathcal{N}(Q_+^T \bar{x}, \Sigma_+)$ is (nondegenerate) Gaussian (hence, density formula holds)
  - $w = Q_0^T \bar{x} \in \mathbf{R}^n$ is not random, called *deterministic component* of x

Linear measurements

linear measurements with noise:

$$y = Ax + v$$

- $x \in \mathbf{R}^n$ is what we want to measure or estimate
- $y \in \mathbf{R}^m$ is measurement
- $A \in \mathbf{R}^{m \times n}$ characterizes sensors or measurements
- v is sensor noise

common assumptions:

- $x \sim \mathcal{N}(\bar{x}, \Sigma_x)$
- $v \sim \mathcal{N}(\bar{v}, \Sigma_v)$
- x and v are independent

- $\mathcal{N}(\bar{x}, \Sigma_x)$ is the *prior distribution* of x (describes initial uncertainty about x)
- $\bar{v}$ is noise *bias* or *offset* (and is usually 0)
- Σ_v is noise *covariance*

thus

$$\begin{bmatrix} x \\ v \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \bar{x} \\ \bar{v} \end{bmatrix}, \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_v \end{bmatrix}\right)$$

using

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

we can write

$$\mathbf{E} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{x} \\ A\bar{x} + \bar{v} \end{bmatrix}$$

and

$$\begin{aligned} \mathbf{E} \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix} \begin{bmatrix} x - \bar{x} \\ y - \bar{y} \end{bmatrix}^T &= \begin{bmatrix} I & 0 \\ A & I \end{bmatrix} \begin{bmatrix} \Sigma_x & 0 \\ 0 & \Sigma_v \end{bmatrix} \begin{bmatrix} I & 0 \\ A & I \end{bmatrix}^T \\ &= \begin{bmatrix} \Sigma_x & \Sigma_x A^T \\ A\Sigma_x & A\Sigma_x A^T + \Sigma_v \end{bmatrix} \end{aligned}$$

covariance of measurement y is $A\Sigma_x A^T + \Sigma_v$

- $A\Sigma_x A^T$ is 'signal covariance'
- Σ_v is 'noise covariance'

Minimum mean-square estimation

suppose $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ are random vectors (not necessarily Gaussian)

we seek to estimate x given y

thus we seek a function $\phi : \mathbf{R}^m \to \mathbf{R}^n$ such that $\hat{x} = \phi(y)$ is near x

one common measure of nearness: mean-square error,

$$\mathbf{E}\,\|\phi(y) - x\|^2$$

minimum mean-square estimator (MMSE) $\phi_{\rm mmse}$ minimizes this quantity

general solution: $\phi_{\rm mmse}(y) = \mathbf{E}(x|y)$, *i.e.*, the conditional expectation of x given y

MMSE for Gaussian vectors

now suppose $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$ are jointly Gaussian:

$$\left[\begin{array}{c} x \\ y \end{array}\right] \sim \mathcal{N}\left(\left[\begin{array}{c} \bar{x} \\ \bar{y} \end{array}\right], \left[\begin{array}{cc} \Sigma_x & \Sigma_{xy} \\ \Sigma_{xy}^T & \Sigma_y \end{array}\right] \right)$$

(after a lot of algebra) the conditional density is

$$p_{x|y}(v|y) = (2\pi)^{-n/2}(\det \Lambda)^{-1/2} \exp\left(-\frac{1}{2}(v-w)^T \Lambda^{-1}(v-w)\right),$$

where

$$\Lambda = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}^T, \quad w = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

hence MMSE estimator ($i.e.$, conditional expectation) is

$$\hat{x} = \phi_{\mathrm{mmse}}(y) = \mathbf{E}(x|y) = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

ϕ_{mmse} is an affine function

MMSE estimation error, $\hat{x} - x$, is a Gaussian random vector

$$\hat{x} - x \;\sim\; \mathcal{N}(0, \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}^T)$$

note that

$$\Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{xy}^T \leq \Sigma_x$$

$i.e.$, covariance of estimation error is always less than prior covariance of x

Best linear unbiased estimator

estimator

$$\hat{x} = \phi_{\mathrm{blu}}(y) = \bar{x} + \Sigma_{xy}\Sigma_y^{-1}(y - \bar{y})$$

makes sense when x, y aren't jointly Gaussian

this estimator

- is *unbiased*, $i.e.$, $\mathbf{E}\,\hat{x} = \mathbf{E}\,x$
- often works well
- is widely used
- has minimum mean square error among all *affine* estimators

sometimes called *best linear unbiased* estimator

MMSE with linear measurements

consider specific case

$$y = Ax + v, \quad x \sim \mathcal{N}(\bar{x}, \Sigma_x), \quad v \sim \mathcal{N}(\bar{v}, \Sigma_v),$$

x, v independent

MMSE of x given y is affine function

$$\hat{x} = \bar{x} + B(y - \bar{y})$$

where $B = \Sigma_x A^T (A\Sigma_x A^T + \Sigma_v)^{-1}$, $\bar{y} = A\bar{x} + \bar{v}$

intepretation:

- $\bar{x}$ is our best prior guess of x (before measurement)
- $y - \bar{y}$ is the discrepancy between what we actually measure (y) and the expected value of what we measure ($\bar{y}$)

- estimator modifies prior guess by B times this discrepancy

- estimator blends prior information with measurement

- B gives *gain* from *observed discrepancy* to *estimate*

- B is small if noise term Σ_v in 'denominator' is large

MMSE error with linear measurements

MMSE estimation error, $\tilde{x} = \hat{x} - x$, is Gaussian with zero mean and covariance

$$\Sigma_{\rm est} = \Sigma_x - \Sigma_x A^T (A\Sigma_x A^T + \Sigma_v)^{-1} A\Sigma_x$$

- $\Sigma_{\rm est} \leq \Sigma_x$, *i.e.*, measurement always decreases uncertainty about x
- difference $\Sigma_x - \Sigma_{\rm est}$ (or some other comparison) gives *value* of measurement y in estimating x
  - $(\Sigma_{\rm est\ \it ii}/\Sigma_{x\ ii})^{1/2}$ gives fractional decrease in uncertainty of x_i due to measurement
  - $(\mathbf{Tr}\,\Sigma_{\rm est}/\,\mathbf{Tr}\,\Sigma)^{1/2}$ gives fractional decrease in uncertainty in x, measured by mean-square error

Estimation error covariance

- error covariance Σ_{est} can be determined *before* measurement y is made!
- to evaluate Σ_{est}, only need to know
  - A (which characterizes sensors)
  - prior covariance of x ($i.e.$, Σ_x)
  - noise covariance ($i.e.$, Σ_v)
- you *do not* need to know the measurement y (or the means $\bar{x}$, $\bar{v}$)
- useful for *experiment design* or *sensor selection*

Information matrix formulas

we can write estimator gain matrix as

$$\begin{aligned} B &= \Sigma_x A^T (A \Sigma_x A^T + \Sigma_v)^{-1} \\ &= \left(A^T \Sigma_v^{-1} A + \Sigma_x^{-1} \right)^{-1} A^T \Sigma_v^{-1} \end{aligned}$$

- $n \times n$ inverse instead of $m \times m$
- Σ_x^{-1}, Σ_v^{-1} sometimes called *information matrices*

corresponding formula for estimator error covariance:

$$\begin{aligned} \Sigma_{\mathrm{est}} &= \Sigma_x - \Sigma_x A^T (A \Sigma_x A^T + \Sigma_v)^{-1} A \Sigma_x \\ &= \left(A^T \Sigma_v^{-1} A + \Sigma_x^{-1} \right)^{-1} \end{aligned}$$

can interpret $\Sigma_{\text{est}}^{-1} = \Sigma_x^{-1} + A^T\Sigma_v^{-1}A$ as:

$$\begin{array}{c}\text{posterior information matrix } (\Sigma_{\text{est}}^{-1}) \\ = \text{prior information matrix } (\Sigma_x^{-1}) \\ + \text{ information added by measurement } (A^T\Sigma_v^{-1}A)\end{array}$$

proof: multiply

$$\Sigma_x A^T (A\Sigma_x A^T + \Sigma_v)^{-1} \stackrel{?}{=} \left(A^T \Sigma_v^{-1} A + \Sigma_x^{-1}\right)^{-1} A^T \Sigma_v^{-1}$$

on left by $(A^T \Sigma_v^{-1} A + \Sigma_x^{-1})$ and on right by $(A\Sigma_x A^T + \Sigma_v)$ to get

$$(A^T \Sigma_v^{-1} A + \Sigma_x^{-1})\Sigma_x A^T \stackrel{?}{=} A^T \Sigma_v^{-1}(A\Sigma_x A^T + \Sigma_v)$$

which is true

Relation to regularized least-squares

suppose $\bar{x} = 0$, $\bar{v} = 0$, $\Sigma_x = \alpha^2 I$, $\Sigma_v = \beta^2 I$

estimator is $\hat{x} = By$ where

$$\begin{aligned} B &= \left(A^T \Sigma_v^{-1} A + \Sigma_x^{-1}\right)^{-1} A^T \Sigma_v^{-1} \\ &= (A^T A + (\beta/\alpha)^2 I)^{-1} A^T \end{aligned}$$

. . . which corresponds to regularized least-squares

MMSE estimate $\hat{x}$ minimizes

$$\|Az - y\|^2 + (\beta/\alpha)^2 \|z\|^2$$

over z

Example

navigation using range measurements to distant beacons

$$y = Ax + v$$

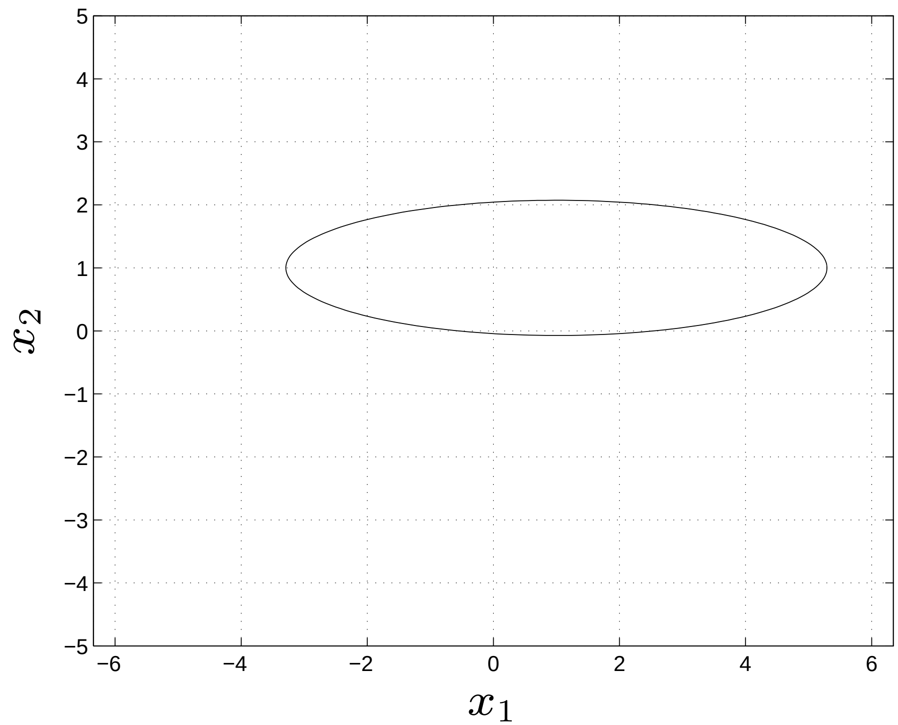
- $x \in \mathbf{R}^2$ is location
- y_i is range measurement to ith beacon
- v_i is range measurement error, IID $\mathcal{N}(0,1)$
- ith row of A is unit vector in direction of ith beacon

prior distribution:

$$x \sim \mathcal{N}(\bar{x}, \Sigma_x), \quad \bar{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \Sigma_x = \begin{bmatrix} 2^2 & 0 \\ 0 & 0.5^2 \end{bmatrix}$$

x_1 has std. dev. 2; x_2 has std. dev. 0.5

90% confidence ellipsoid for prior distribution
$\{ \ x \mid (x - \bar{x})^T \Sigma_x^{-1} (x - \bar{x}) \le 4.6 \ \}$:



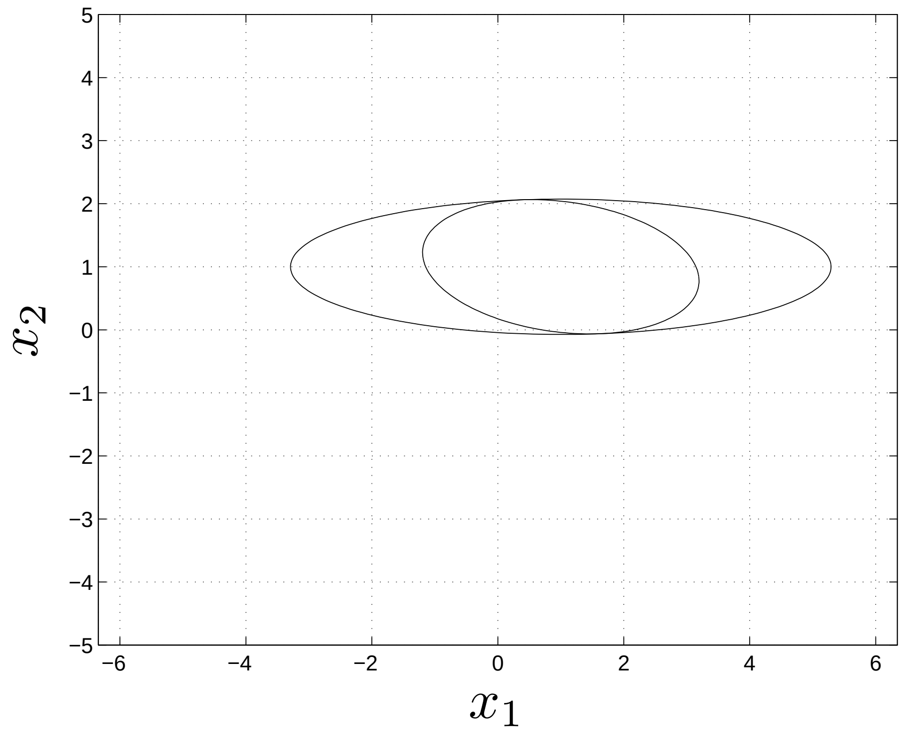
Case 1: one measurement, with beacon at angle 30°

fewer measurements than variables, so combining prior information with measurement is critical

resulting estimation error covariance:

$$\Sigma_{\rm est} = \left[\begin{array}{rr} 1.046 & -0.107 \\ -0.107 & 0.246 \end{array}\right]$$

90% confidence ellipsoid for estimate $\hat{x}$: (and 90% confidence ellipsoid for x)



interpretation: measurement

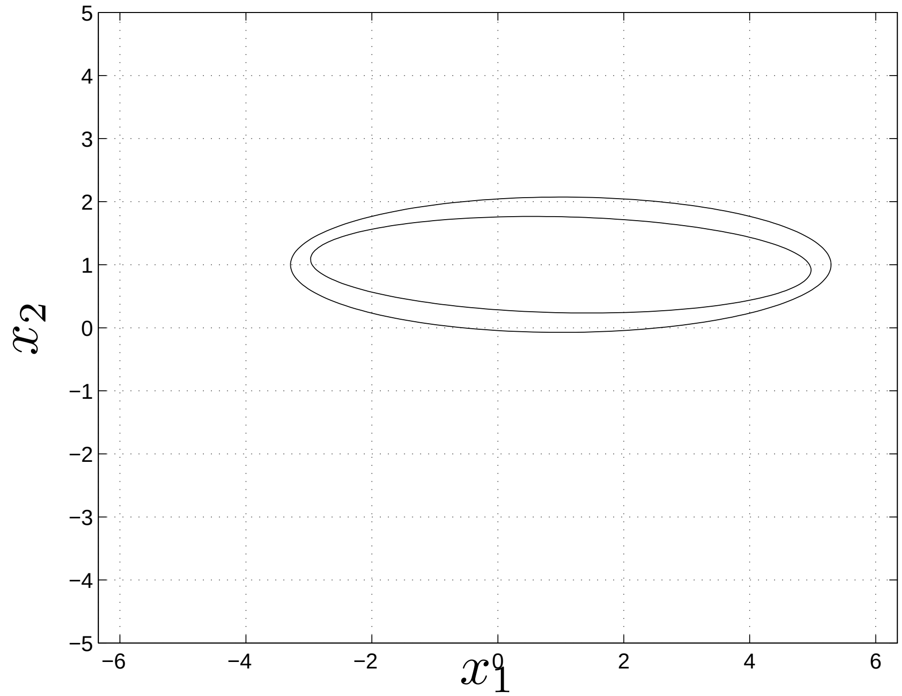
- yields essentially no reduction in uncertainty in x_2
- reduces uncertainty in x_1 by a factor about two

Case 2: 4 measurements, with beacon angles $80°,\ 85°,\ 90°,\ 95°$

resulting estimation error covariance:

$$\Sigma_{\rm est} = \begin{bmatrix} 3.429 & -0.074 \\ -0.074 & 0.127 \end{bmatrix}$$

90% confidence ellipsoid for estimate $\hat{x}$: (and 90% confidence ellipsoid for x)



interpretation: measurement yields

- little reduction in uncertainty in x_1
- small reduction in uncertainty in x_2

Lecture 7
The Kalman filter

- Linear system driven by stochastic process
- Statistical steady-state
- Linear Gauss-Markov model
- Kalman filter
- Steady-state Kalman filter

Linear system driven by stochastic process

we consider linear dynamical system $x(t+1) = Ax(t) + Bu(t)$, with $x(0)$ and $u(0),\ u(1), \ldots$ random variables

we'll use notation

$$\bar{x}(t) = \mathbf{E}\, x(t), \qquad \Sigma_x(t) = \mathbf{E}(x(t) - \bar{x}(t))(x(t) - \bar{x}(t))^T$$

and similarly for $\bar{u}(t)$, $\Sigma_u(t)$

taking expectation of $x(t+1) = Ax(t) + Bu(t)$ we have

$$\bar{x}(t+1) = A\bar{x}(t) + B\bar{u}(t)$$

i.e., the means propagate by the same linear dynamical system now let's consider the covariance

$$x(t+1) - \bar{x}(t+1) = A(x(t) - \bar{x}(t)) + B(u(t) - \bar{u}(t))$$

and so

$$\begin{aligned} \Sigma_x(t+1) &= \mathbf{E}\left(A(x(t) - \bar{x}(t)) + B(u(t) - \bar{u}(t))\right) \cdot \\ &\quad \cdot \left(A(x(t) - \bar{x}(t)) + B(u(t) - \bar{u}(t))\right)^T \\ &= A\Sigma_x(t)A^T + B\Sigma_u(t)B^T + A\Sigma_{xu}(t)B^T + B\Sigma_{ux}(t)A^T \end{aligned}$$

where

$$\Sigma_{xu}(t) = \Sigma_{ux}(t)^T = \mathbf{E}(x(t) - \bar{x}(t))(u(t) - \bar{u}(t))^T$$

thus, the covariance $\Sigma_x(t)$ satisfies another, Lyapunov-like linear dynamical system, driven by Σ_{xu} and Σ_u

consider special case $\Sigma_{xu}(t) = 0$, $i.e.$, x and u are uncorrelated, so we have Lyapunov iteration

$$\Sigma_x(t+1) = A\Sigma_x(t)A^T + B\Sigma_u(t)B^T,$$

which is stable if and only if A is stable

if A is stable and $\Sigma_u(t)$ is constant, $\Sigma_x(t)$ converges to Σ_x, called the *steady-state covariance*, which satisfies Lyapunov equation

$$\Sigma_x = A\Sigma_x A^T + B\Sigma_u B^T$$

thus, we can calculate the steady-state covariance of x exactly, by solving a Lyapunov equation

(useful for starting simulations in statistical steady-state)

Example

we consider $x(t+1) = Ax(t) + w(t)$, with

$$A = \begin{bmatrix} 0.6 & -0.8 \\ 0.7 & 0.6 \end{bmatrix},$$

where $w(t)$ are IID $\mathcal{N}(0, I)$

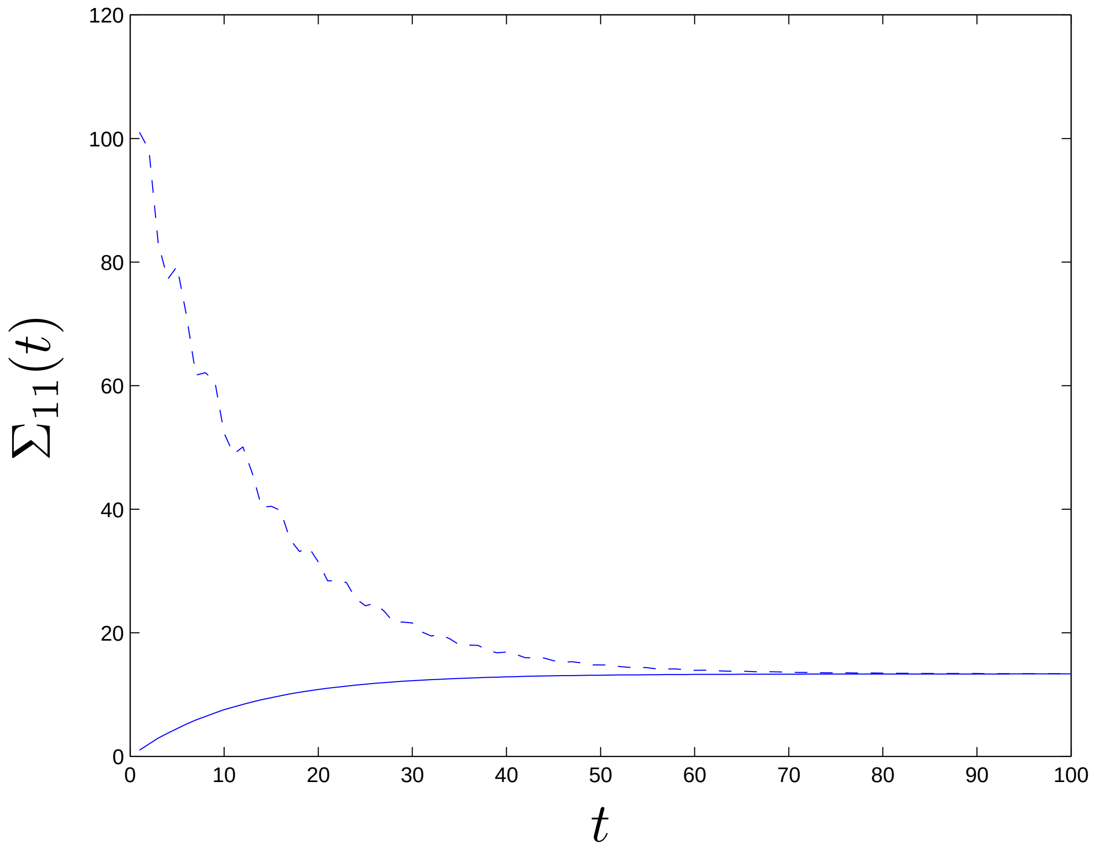
eigenvalues of A are $0.6 \pm 0.75j$, with magnitude 0.96, so A is stable

we solve Lyapunov equation to find steady-state covariance

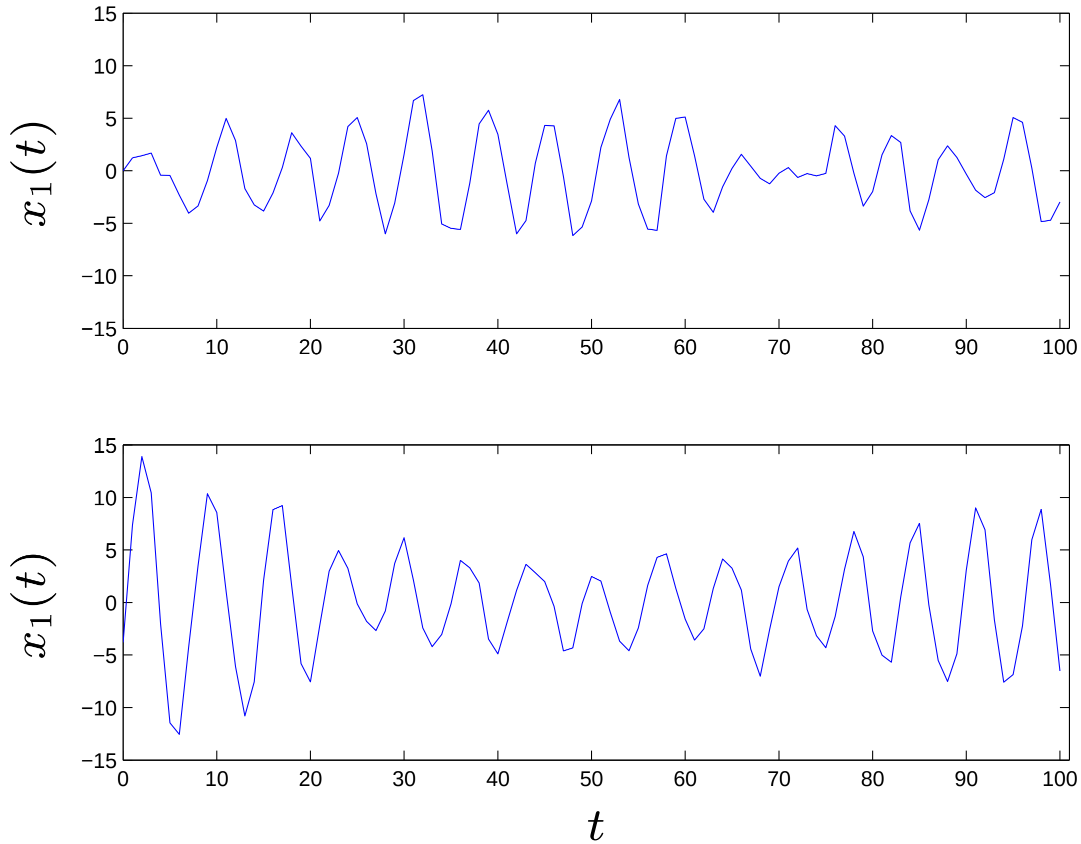
$$\Sigma_x = \begin{bmatrix} 13.35 & -0.03 \\ -0.03 & 11.75 \end{bmatrix}$$

covariance of $x(t)$ converges to Σ_x no matter its initial value two initial state distributions: $\Sigma_x(0) = 0$, $\Sigma_x(0) = 10^2 I$

plot shows $\Sigma_{11}(t)$ for the two cases



$x_1(t)$ for one realization from each case:

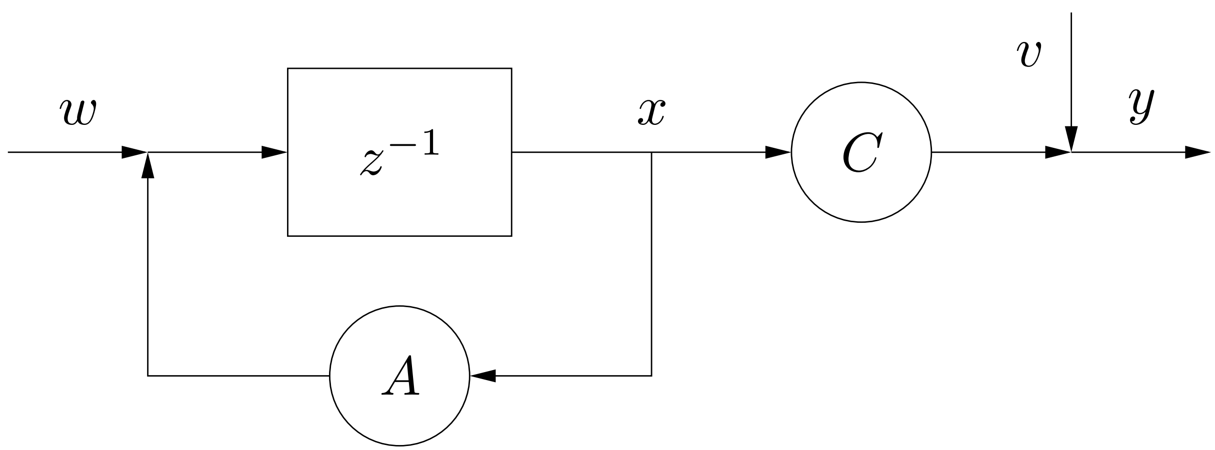


Linear Gauss-Markov model

we consider linear dynamical system

$$x(t+1) = Ax(t) + w(t), \qquad y(t) = Cx(t) + v(t)$$

- $x(t) \in \mathbf{R}^n$ is the state; $y(t) \in \mathbf{R}^p$ is the observed output
- $w(t) \in \mathbf{R}^n$ is called *process noise* or *state noise*
- $v(t) \in \mathbf{R}^p$ is called *measurement noise*



Statistical assumptions

- $x(0)$, $w(0), w(1), \ldots$, and $v(0), v(1), \ldots$ are jointly Gaussian and independent
- $w(t)$ are IID with $\mathbf{E}\, w(t) = 0$, $\mathbf{E}\, w(t)w(t)^T = W$
- $v(t)$ are IID with $\mathbf{E}\, v(t) = 0$, $\mathbf{E}\, v(t)v(t)^T = V$
- $\mathbf{E}\, x(0) = \bar{x}_0$, $\mathbf{E}(x(0) - \bar{x}_0)(x(0) - \bar{x}_0)^T = \Sigma_0$

(it's not hard to extend to case where $w(t)$, $v(t)$ are not zero mean)

we'll denote $X(t) = (x(0), \ldots, x(t))$, etc.

since $X(t)$ and $Y(t)$ are linear functions of $x(0)$, $W(t)$, and $V(t)$, we conclude they are all jointly Gaussian (*i.e.*, the process $x,\ w,\ v,\ y$ is Gaussian)

Statistical properties

- sensor noise v independent of x
- $w(t)$ is independent of $x(0), \ldots, x(t)$ and $y(0), \ldots, y(t)$
- *Markov property:* the process x is Markov, $i.e.$,

$$x(t)|x(0), \ldots, x(t-1) = x(t)|x(t-1)$$

roughly speaking: if you know $x(t-1)$, then knowledge of $x(t-2), \ldots, x(0)$ doesn't give any more information about $x(t)$

Mean and covariance of Gauss-Markov process

mean satisfies $\bar{x}(t+1) = A\bar{x}(t)$, $\bar{x}(0) = \bar{x}_0$, so $\bar{x}(t) = A^t\bar{x}_0$

covariance satisfies

$$\Sigma_x(t+1) = A\Sigma_x(t)A^T + W$$

if A is stable, $\Sigma_x(t)$ converges to steady-state covariance Σ_x, which satisfies Lyapunov equation

$$\Sigma_x = A\Sigma_x A^T + W$$

Conditioning on observed output

we use the notation

$$\begin{aligned}\hat{x}(t|s) &= \mathbf{E}(x(t)|y(0),\dots y(s)),\\ \Sigma_{t|s} &= \mathbf{E}(x(t)-\hat{x}(t|s))(x(t)-\hat{x}(t|s))^T\end{aligned}$$

- the random variable $x(t)|y(0),\dots,y(s)$ is Gaussian, with mean $\hat{x}(t|s)$ and covariance $\Sigma_{t|s}$
- $\hat{x}(t|s)$ is the minimum mean-square error estimate of $x(t)$, based on $y(0),\dots,y(s)$
- $\Sigma_{t|s}$ is the covariance of the error of the estimate $\hat{x}(t|s)$

State estimation

we focus on two state estimation problems:

- finding $\hat{x}(t|t)$, $i.e.$, estimating the current state, based on the current and past observed outputs
- finding $\hat{x}(t+1|t)$, $i.e.$, predicting the next state, based on the current and past observed outputs

since $x(t), Y(t)$ are jointly Gaussian, we can use the standard formula to find $\hat{x}(t|t)$ (and similarly for $\hat{x}(t+1|t)$)

$$\hat{x}(t|t) = \bar{x}(t) + \Sigma_{x(t)Y(t)}\Sigma_{Y(t)}^{-1}(Y(t) - \bar{Y}(t))$$

the inverse in the formula, $\Sigma_{Y(t)}^{-1}$, is size $pt \times pt$, which grows with t

the *Kalman filter* is a clever method for computing $\hat{x}(t|t)$ and $\hat{x}(t+1|t)$ recursively

Measurement update

let's find $\hat{x}(t|t)$ and $\Sigma_{t|t}$ in terms of $\hat{x}(t|t-1)$ and $\Sigma_{t|t-1}$

start with $y(t) = Cx(t) + v(t)$, and condition on $Y(t-1)$:

$$y(t)|Y(t-1) = Cx(t)|Y(t-1) + v(t)|Y(t-1) = Cx(t)|Y(t-1) + v(t)$$

since $v(t)$ and $Y(t-1)$ are independent

so $x(t)|Y(t-1)$ and $y(t)|Y(t-1)$ are jointly Gaussian with mean and covariance

$$\left[\begin{array}{c} \hat{x}(t|t-1) \\ C\hat{x}(t|t-1) \end{array}\right], \qquad \left[\begin{array}{cc} \Sigma_{t|t-1} & \Sigma_{t|t-1}C^T \\ C\Sigma_{t|t-1} & C\Sigma_{t|t-1}C^T + V \end{array}\right]$$

now use standard formula to get mean and covariance of

$$(x(t)|Y(t-1))|(y(t)|Y(t-1)),$$

which is exactly the same as $x(t)|Y(t)$:

$$\begin{aligned}
\hat{x}(t|t) &= \hat{x}(t|t-1) + \Sigma_{t|t-1}C^T\left(C\Sigma_{t|t-1}C^T+V\right)^{-1}(y(t)-C\hat{x}(t|t-1)) \\
\Sigma_{t|t} &= \Sigma_{t|t-1} - \Sigma_{t|t-1}C^T\left(C\Sigma_{t|t-1}C^T+V\right)^{-1}C\Sigma_{t|t-1}
\end{aligned}$$

this gives us $\hat{x}(t|t)$ and $\Sigma_{t|t}$ in terms of $\hat{x}(t|t-1)$ and $\Sigma_{t|t-1}$

this is called the *measurement update* since it gives our updated estimate of $x(t)$ based on the measurement $y(t)$ becoming available

Time update

now let's increment time, using $x(t+1) = Ax(t) + w(t)$

condition on $Y(t)$ to get

$$\begin{aligned} x(t+1)|Y(t) &= Ax(t)|Y(t) + w(t)|Y(t) \\ &= Ax(t)|Y(t) + w(t) \end{aligned}$$

since $w(t)$ is independent of $Y(t)$

therefore we have $\hat{x}(t+1|t) = A\hat{x}(t|t)$ and

$$\begin{aligned} \Sigma_{t+1|t} &= \mathbf{E}(\hat{x}(t+1|t) - x(t+1))(\hat{x}(t+1|t) - x(t+1))^T \\ &= \mathbf{E}(A\hat{x}(t|t) - Ax(t) - w(t))(A\hat{x}(t|t) - Ax(t) - w(t))^T \\ &= A\Sigma_{t|t}A^T + W \end{aligned}$$

Kalman filter

measurement and time updates together give a recursive solution

start with prior mean and covariance, $\hat{x}(0|-1) = \bar{x}_0$, $\Sigma(0|-1) = \Sigma_0$

apply the measurement update

$$\begin{aligned}
\hat{x}(t|t) &= \hat{x}(t|t-1) + \Sigma_{t|t-1}C^T \left(C\Sigma_{t|t-1}C^T + V\right)^{-1} (y(t) - C\hat{x}(t|t-1)) \\
\Sigma_{t|t} &= \Sigma_{t|t-1} - \Sigma_{t|t-1}C^T \left(C\Sigma_{t|t-1}C^T + V\right)^{-1} C\Sigma_{t|t-1}
\end{aligned}$$

to get $\hat{x}(0|0)$ and $\Sigma_{0|0}$; then apply time update

$$\hat{x}(t+1|t) = A\hat{x}(t|t), \qquad \Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W$$

to get $\hat{x}(1|0)$ and $\Sigma_{1|0}$

now, repeat measurement and time updates . . .

Riccati recursion

to lighten notation, we'll use $\hat{x}(t) = \hat{x}(t|t-1)$ and $\hat{\Sigma}_t = \Sigma_{t|t-1}$

we can express measurement and time updates for $\hat{\Sigma}$ as

$$\hat{\Sigma}_{t+1} = A\hat{\Sigma}_t A^T + W - A\hat{\Sigma}_t C^T (C\hat{\Sigma}_t C^T + V)^{-1} C\hat{\Sigma}_t A^T$$

which is a Riccati recursion, with initial condition $\hat{\Sigma}_0 = \Sigma_0$

- $\hat{\Sigma}_t$ can be computed *before any observations are made*
- thus, we can calculate the estimation error covariance *before* we get any observed data

Comparison with LQR

in LQR,

- Riccati recursion for $P(t)$ (which determines the minimum cost to go from a point at time t) runs *backward* in time
- we can compute cost-to-go before knowing $x(t)$

in Kalman filter,

- Riccati recursion for $\hat{\Sigma}_t$ (which is the state prediction error covariance at time t) runs *forward* in time
- we can compute $\hat{\Sigma}_t$ before we actually get any observations

Observer form

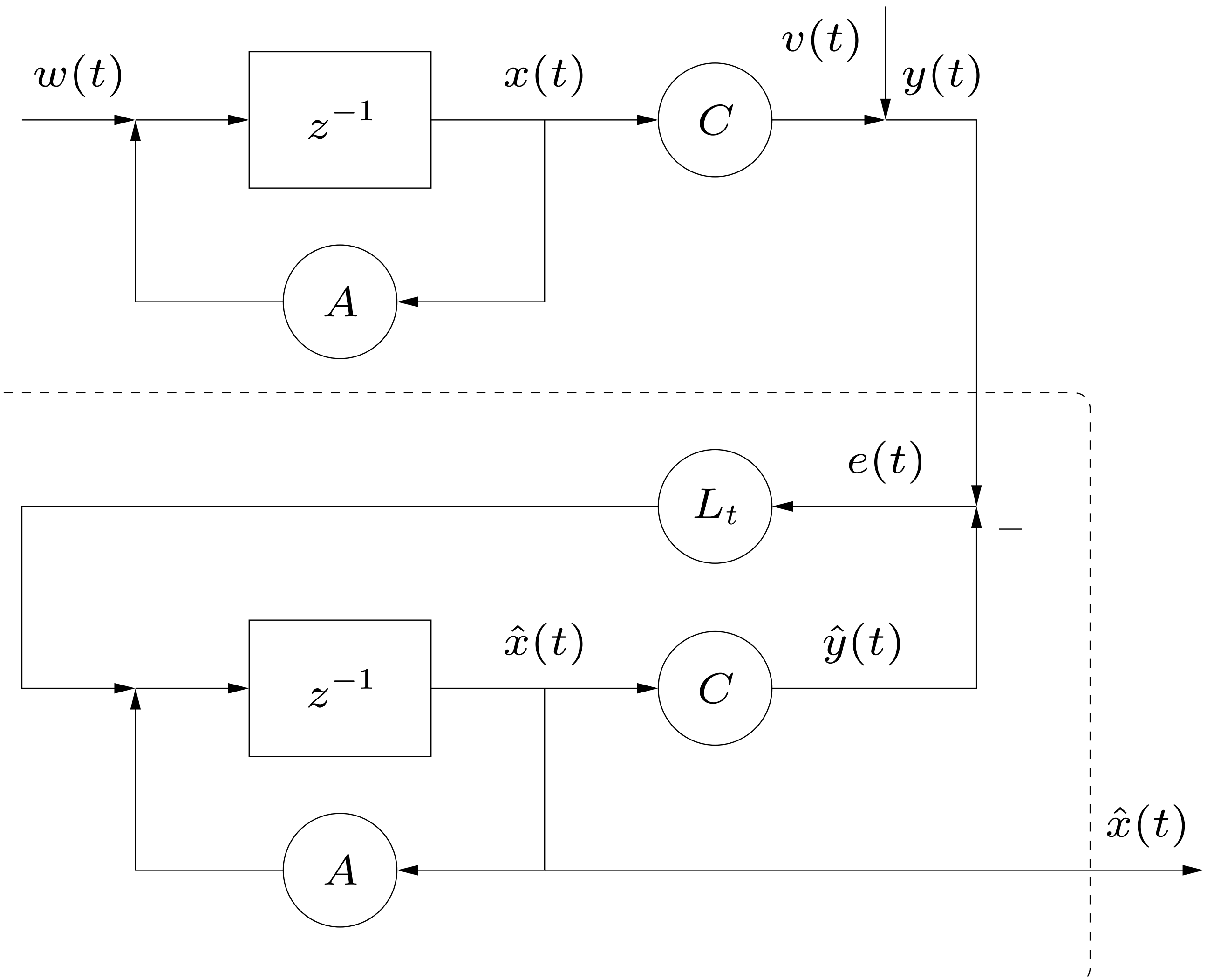
we can express KF as

$$\begin{aligned}\hat{x}(t+1) &= A\hat{x}(t) + A\hat{\Sigma}_t C^T (C\hat{\Sigma}_t C^T + V)^{-1}(y(t) - C\hat{x}(t)) \\ &= A\hat{x}(t) + L_t(y(t) - \hat{y}(t))\end{aligned}$$

where $L_t = A\hat{\Sigma}_t C^T (C\hat{\Sigma}_t C^T + V)^{-1}$ is the *observer gain*, and $\hat{y}(t)$ is $\hat{y}(t|t-1)$

- $\hat{y}(t)$ is our output prediction, $i.e.$, our estimate of $y(t)$ based on $y(0), \ldots, y(t-1)$
- $e(t) = y(t) - \hat{y}(t)$ is our output prediction error
- $A\hat{x}(t)$ is our prediction of $x(t+1)$ based on $y(0), \ldots, y(t-1)$
- our estimate of $x(t+1)$ is the prediction based on $y(0), \ldots, y(t-1)$, plus a linear function of the output prediction error

Kalman filter block diagram



Steady-state Kalman filter

as in LQR, Riccati recursion for $\hat{\Sigma}_t$ converges to steady-state value $\hat{\Sigma}$, provided (C, A) is observable and (A, W) is controllable

$\hat{\Sigma}$ gives steady-state error covariance for estimating $x(t+1)$ given $y(0), \ldots, y(t)$

note that state prediction error covariance converges, even if system is unstable

$\hat{\Sigma}$ satisfies ARE

$$\hat{\Sigma} = A\hat{\Sigma}A^T + W - A\hat{\Sigma}C^T(C\hat{\Sigma}C^T + V)^{-1}C\hat{\Sigma}A^T$$

(which can be solved directly)

steady-state filter is a time-invariant observer:

$$\hat{x}(t+1) = A\hat{x}(t) + L(y(t) - \hat{y}(t)), \qquad \hat{y}(t) = C\hat{x}(t)$$

where $L = A\hat{\Sigma}C^T(C\hat{\Sigma}C^T + V)^{-1}$

define state estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$, so

$$y(t) - \hat{y}(t) = Cx(t) + v(t) - C\hat{x}(t) = C\tilde{x}(t) + v(t)$$

and

$$\begin{aligned}
\tilde{x}(t+1) &= x(t+1) - \hat{x}(t+1) \\
&= Ax(t) + w(t) - A\hat{x}(t) - L(C\tilde{x}(t) + v(t)) \\
&= (A - LC)\tilde{x}(t) + w(t) - Lv(t)
\end{aligned}$$

thus, the estimation error propagates according to a linear system, with closed-loop dynamics $A - LC$, driven by the process $w(t) - LCv(t)$, which is IID zero mean and covariance $W + LVL^T$

provided A, W is controllable and C, A is observable, $A - LC$ is stable

Example

system is

$$x(t+1) = Ax(t) + w(t), \qquad y(t) = Cx(t) + v(t)$$

with $x(t) \in \mathbf{R}^6$, $y(t) \in \mathbf{R}$

we'll take $\mathbf{E}\,x(0) = 0$, $\mathbf{E}\,x(0)x(0)^T = \Sigma_0 = 5^2 I$; $W = (1.5)^2 I$, $V = 1$

eigenvalues of A:

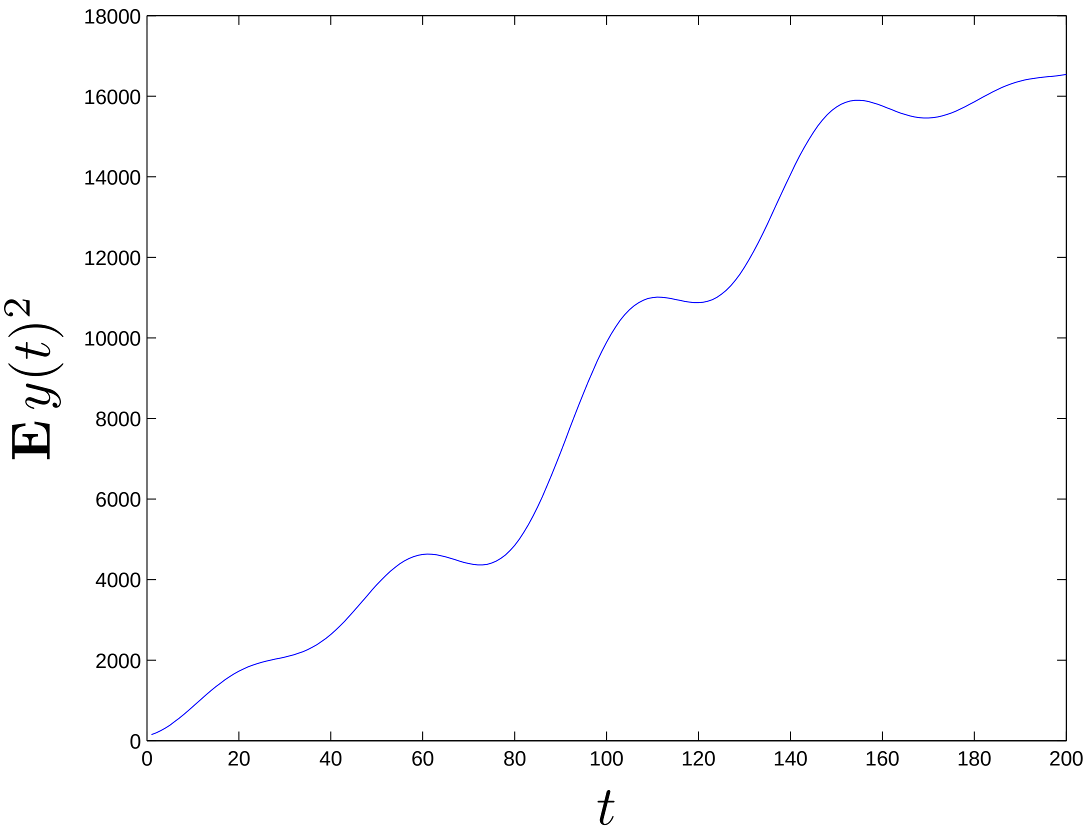
$$0.9973 \pm 0.0730j, \qquad 0.9995 \pm 0.0324j, \qquad 0.9941 \pm 0.1081j$$

(which have magnitude one)

goal: predict $y(t+1)$ based on $y(0), \ldots, y(t)$

first let's find variance of $y(t)$ versus t, using Lyapunov recursion

$$\mathbf{E}\, y(t)^2 = C\Sigma_x(t)C^T + V, \qquad \Sigma_x(t+1) = A\Sigma_x(t)A^T + W, \qquad \Sigma_x(0) = \Sigma_0$$

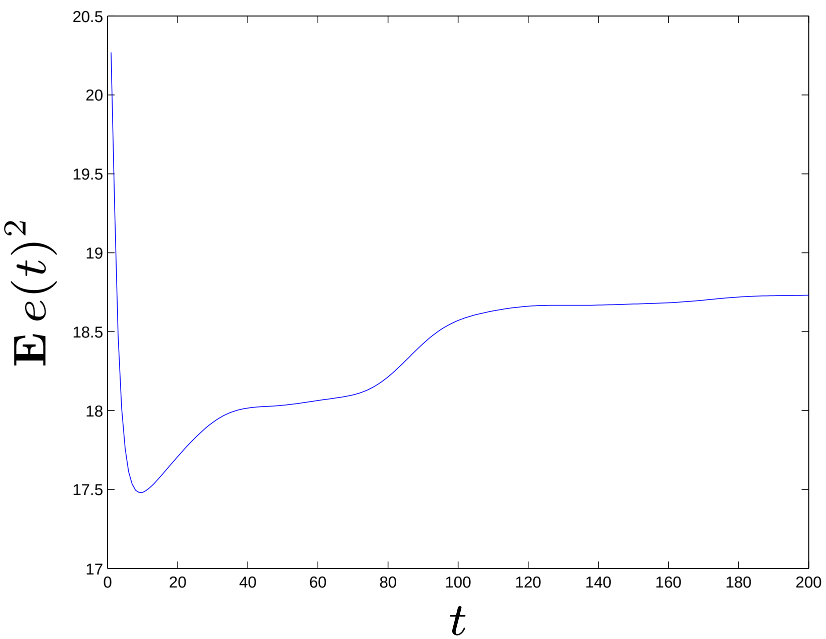


now, let's plot the prediction error variance versus t,

$$\mathbf{E}\, e(t)^2 = \mathbf{E}(\hat{y}(t) - y(t))^2 = C\hat{\Sigma}_t C^T + V,$$

where $\hat{\Sigma}_t$ satisfies Riccati recursion

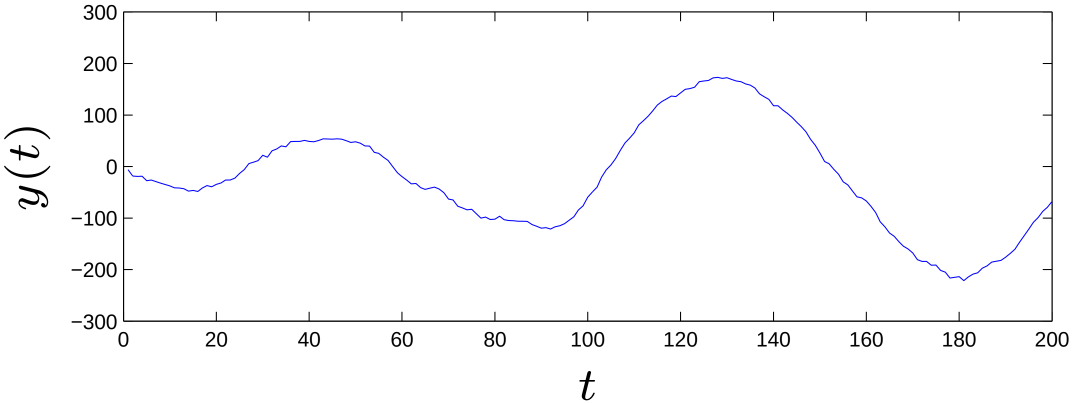
$$\hat{\Sigma}_{t+1} = A\hat{\Sigma}_t A^T + W - A\hat{\Sigma}_t C^T (C\hat{\Sigma}_t C^T + V)^{-1} C\hat{\Sigma}_t A^T, \qquad \hat{\Sigma}_{-1} = \Sigma_0$$

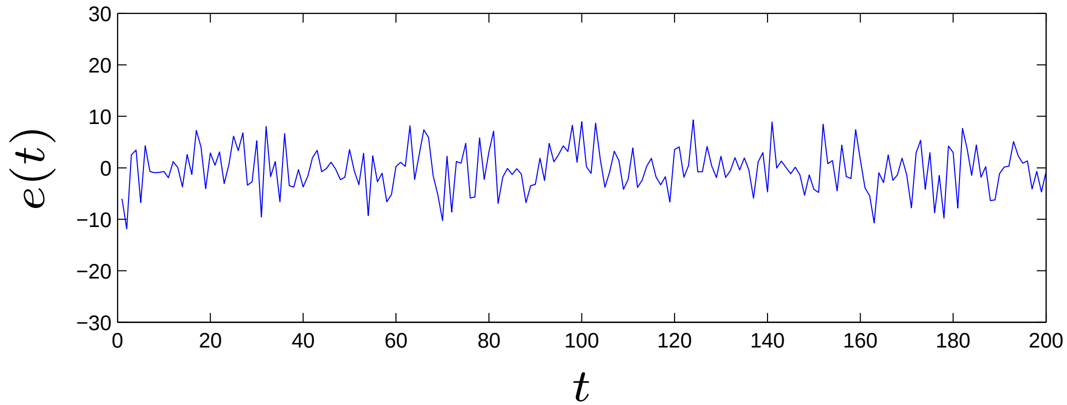


prediction error variance converges to steady-state value 18.7

now let's try the Kalman filter on a realization $y(t)$

top plot shows $y(t)$; bottom plot shows $e(t)$ (on different vertical scale)





Lecture 8
The Extended Kalman filter

- Nonlinear filtering
- Extended Kalman filter
- Linearization and random variables

Nonlinear filtering

- nonlinear Markov model:

$$x(t+1) = f(x(t), w(t)), \qquad y(t) = g(x(t), v(t))$$

  - f is (possibly nonlinear) dynamics function
  - g is (possibly nonlinear) measurement or output function
  - $w(0), w(1), \ldots, v(0), v(1), \ldots$ are independent
  - even if w, v Gaussian, x and y need not be

- nonlinear filtering problem: find, *e.g.*,

$$\hat{x}(t|t-1) = \mathbf{E}(x(t)|y(0), \ldots, y(t-1)), \qquad \hat{x}(t|t) = \mathbf{E}(x(t)|y(0), \ldots, y(t))$$

- general nonlinear filtering solution involves a PDE, and is not practical

Extended Kalman filter

- extended Kalman filter (EKF) is *heuristic* for nonlinear filtering problem
- often works well (when tuned properly), but sometimes not
- widely used in practice
- based on
  - linearizing dynamics and output functions at current estimate
  - propagating an approximation of the conditional expectation and covariance

Linearization and random variables

- consider $\phi : \mathbf{R}^n \to \mathbf{R}^m$
- suppose $\mathbf{E}\, x = \bar{x}$, $\mathbf{E}(x - \bar{x})(x - \bar{x})^T = \Sigma_x$, and $y = \phi(x)$
- if Σ_x is small, ϕ is not too nonlinear,

$$y \approx \tilde{y} = \phi(\bar{x}) + D\phi(\bar{x})(x - \bar{x})$$

$$\tilde{y} \sim \mathcal{N}(\phi(\bar{x}), D\phi(\bar{x})\Sigma_x D\phi(\bar{x})^T)$$

- gives *approximation* for mean and covariance of nonlinear function of random variable:

$$\bar{y} \approx \phi(\bar{x}), \qquad \Sigma_y \approx D\phi(\bar{x})\Sigma_x D\phi(\bar{x})^T$$

- if Σ_x is not small compared to 'curvature' of ϕ, these estimates are poor

- a good estimate can be found by Monte Carlo simulation:

$$\begin{aligned}
\bar{y} &\approx \bar{y}^{\mathrm{mc}} = \frac{1}{N}\sum_{i=1}^{N}\phi(x^{(i)}) \\
\Sigma_y &\approx \frac{1}{N}\sum_{i=1}^{N}\left(\phi(x^{(i)}) - \bar{y}^{\mathrm{mc}}\right)\left(\phi(x^{(i)}) - \bar{y}^{\mathrm{mc}}\right)^T
\end{aligned}$$

  where $x^{(1)}, \ldots, x^{(N)}$ are samples from the distribution of x, and N is large

- another method: use Monte Carlo formulas, with a small number of nonrandom samples chosen as 'typical', $e.g.$, the 90% confidence ellipsoid semi-axis endpoints

$$x^{(i)} = \bar{x} \pm \beta v_i, \qquad \Sigma_x = V\Lambda V^T$$

Example

$x \sim \mathcal{N}(0,1)$, $y = \exp(x)$

(for this case we can compute mean and variance of y exactly)

	$\bar{y}$	σ_y
exact values	$e^{1/2} = 1.649$	$\sqrt{e^2 - e} = 2.161$
linearization	1.000	1.000
Monte Carlo ($N = 10$)	1.385	1.068
Monte Carlo ($N = 100$)	1.430	1.776
Sigma points ($x = \bar{x},\ \bar{x} \pm 1.5\sigma_x$)	1.902	2.268

Extended Kalman filter

- *initialization:* $\hat{x}(0|-1) = \bar{x}_0$, $\Sigma(0|-1) = \Sigma_0$
- *measurement update*
  - linearize output function at $x = \hat{x}(t|t-1)$:

$$\begin{aligned} C &= \frac{\partial g}{\partial x}(\hat{x}(t|t-1), 0) \\ V &= \frac{\partial g}{\partial v}(\hat{x}(t|t-1), 0)\Sigma_v \frac{\partial g}{\partial v}(\hat{x}(t|t-1), 0)^T \end{aligned}$$

  - measurement update based on linearization

$$\begin{aligned} \hat{x}(t|t) &= \hat{x}(t|t-1) + \Sigma_{t|t-1}C^T\left(C\Sigma_{t|t-1}C^T + V\right)^{-1}\ldots \\ &\quad \ldots(y(t) - g(\hat{x}(t|t-1), 0)) \\ \Sigma_{t|t} &= \Sigma_{t|t-1} - \Sigma_{t|t-1}C^T\left(C\Sigma_{t|t-1}C^T + V\right)^{-1}C\Sigma_{t|t-1} \end{aligned}$$

- *time update*
  - linearize dynamics function at $x = \hat{x}(t|t)$:

$$\begin{aligned} A &= \frac{\partial f}{\partial x}(\hat{x}(t|t), 0) \\ W &= \frac{\partial f}{\partial w}(\hat{x}(t|t), 0)\Sigma_w \frac{\partial f}{\partial w}(\hat{x}(t|t), 0)^T \end{aligned}$$

  - time update based on linearization

$$\hat{x}(t+1|t) = f(\hat{x}(t|t), 0), \qquad \Sigma_{t+1|t} = A\Sigma_{t|t}A^T + W$$

- replacing linearization with Monte Carlo yields *particle filter*
- replacing linearization with sigma-point estimates yields *unscented Kalman filter* (UKF)

Example

- $p(t)$, $u(t) \in \mathbf{R}^2$ are position and velocity of vehicle, with $(p(0), u(0)) \sim \mathcal{N}(0, I)$

- vehicle dynamics:

$$p(t+1) = p(t) + 0.1u(t), \qquad u(t+1) = \begin{bmatrix} 0.85 & 0.15 \\ -0.1 & 0.85 \end{bmatrix} u(t) + w(t)$$

  $w(t)$ are IID $\mathcal{N}(0, I)$

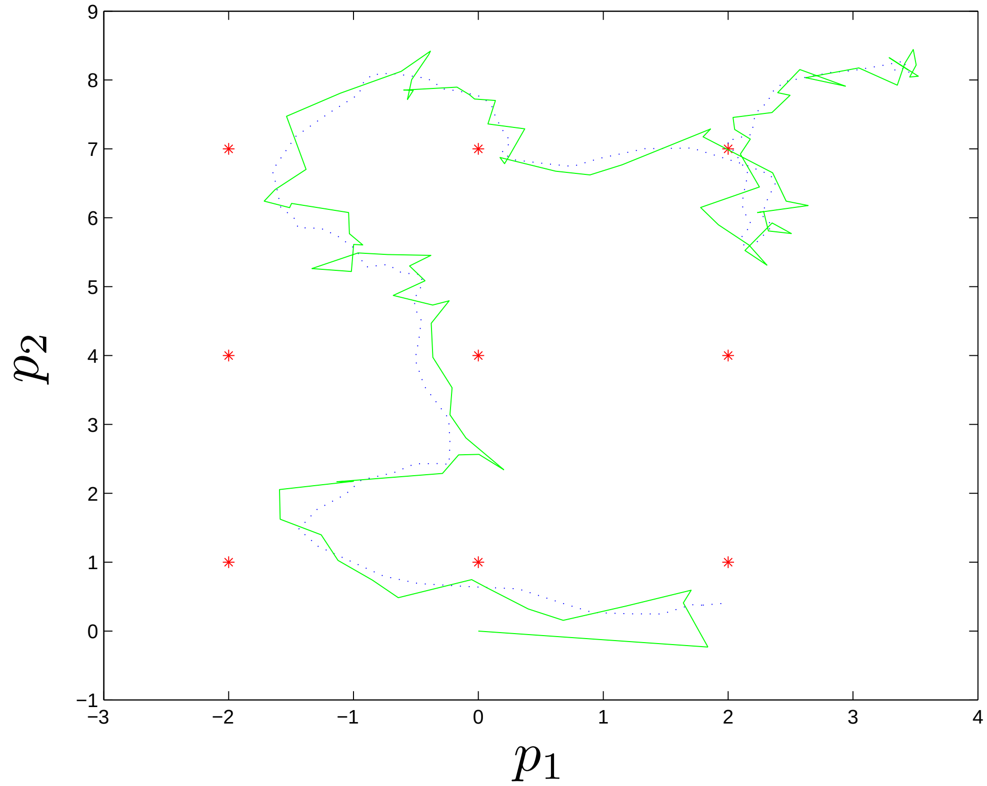
- measurements: noisy measurements of distance to 9 points $p_i \in \mathbf{R}^2$

$$y_i(t) = \|p(t) - p_i\| + v_i(t), \quad i = 1, \ldots, 9,$$

  $v_i(t)$ are IID $\mathcal{N}(0, 0.3^2)$

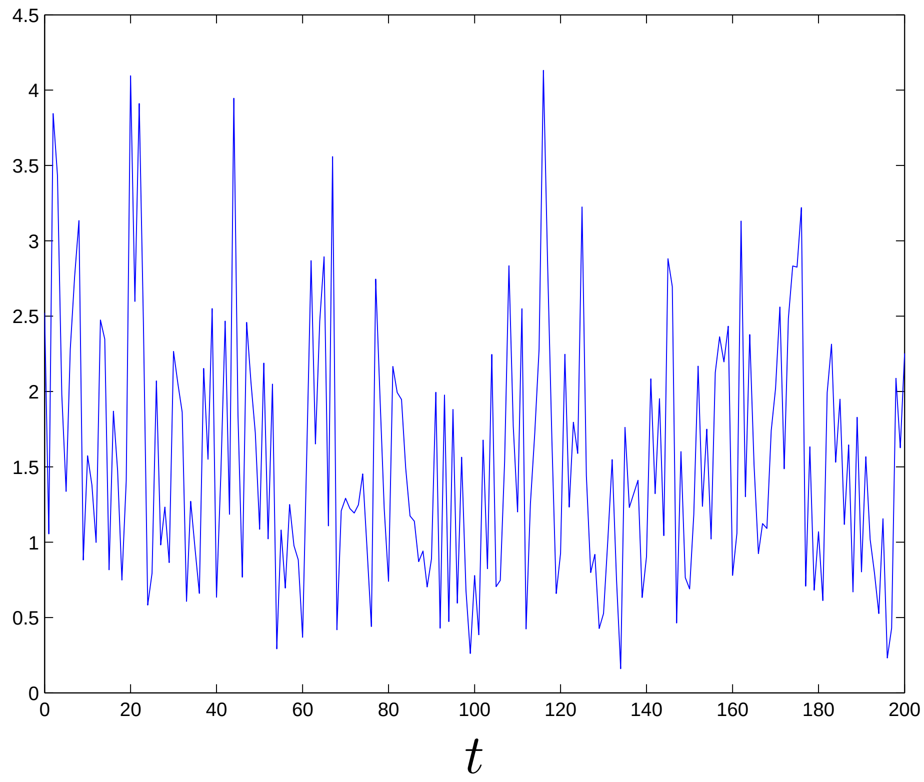
EKF results

- EKF initialized with $\hat{x}(0|-1) = 0$, $\Sigma(0|-1) = I$, where $x = (p, u)$
- p_i shown as stars; $p(t)$ as dotted curve; $\hat{p}(t|t)$ as solid curve



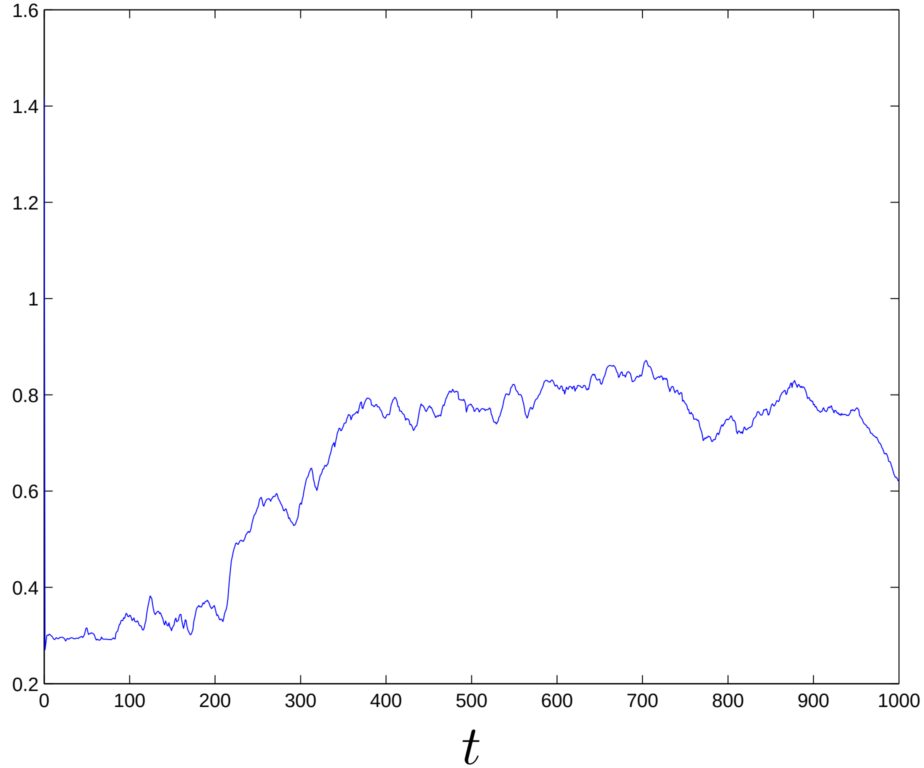
Current position estimation error

$\|\hat{p}(t|t) - p(t)\|$ versus t



Current position estimation predicted error

$(\Sigma(t|t)_{11} + \Sigma(t|t)_{22})^{1/2}$ versus t



Lecture 9
Invariant sets, conservation, and dissipation

- invariant sets
- conserved quantities
- dissipated quantities
- derivative along trajectory
- discrete-time case

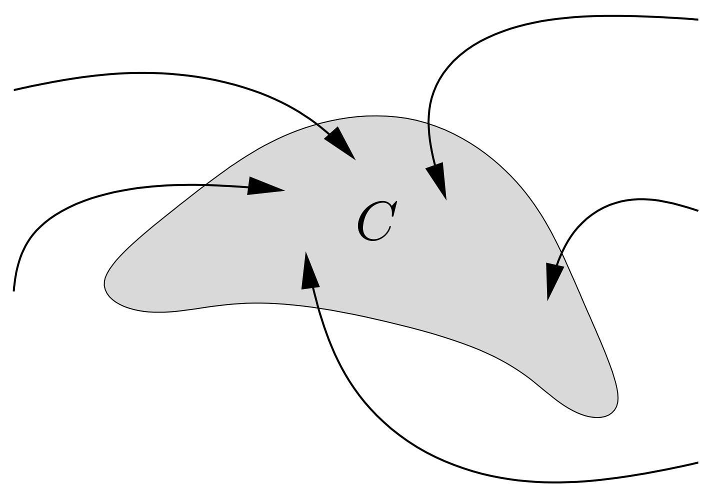
Invariant sets

we consider autonomous, time-invariant nonlinear system $\dot{x} = f(x)$

a set $C \subseteq \mathbf{R}^n$ is *invariant* (w.r.t. system, or f) if for every trajectory x,

$$x(t) \in C \implies x(\tau) \in C \text{ for all } \tau \geq t$$

- if trajectory enters C, or starts in C, it stays in C
- trajectories can cross *into* boundary of C, but never *out* of C



Examples of invariant sets

general examples:

- $\{x_0\}$, where $f(x_0) = 0$ ($i.e.$, x_0 is an equilibrium point)
- any trajectory or union of trajectories, $e.g.$,
  $\{x(t) \mid x(0) \in D,\ t \geq 0,\ \dot{x} = f(x)\}$

more specific examples:

- $\dot{x} = Ax$, $C = \mathrm{span}\{v_1, \ldots, v_k\}$, where $Av_i = \lambda_i v_i$
- $\dot{x} = Ax$, $C = \{z \mid 0 \leq w^T z \leq a\}$, where $w^T A = \lambda w^T$, $\lambda \leq 0$

Invariance of nonnegative orthant

when is nonnegative orthant $\mathbf{R}_+^n$ invariant for $\dot{x} = Ax$?
($i.e.$, when do nonnegative trajectories always stay nonnegative?)

answer: if and only if $A_{ij} \geq 0$ for $i \neq j$

first assume $A_{ij} \geq 0$ for $i \neq j$, and $x(0) \in \mathbf{R}_+^n$; we'll show that $x(t) \in \mathbf{R}_+^n$ for $t \geq 0$

$$x(t) = e^{tA}x(0) = \lim_{k \to \infty} \left(I + (t/k)A\right)^k x(0)$$

for k large enough the matrix $I + (t/k)A$ has all nonnegative entries, so $\left(I + (t/k)A\right)^k x(0)$ has all nonnegative entries

hence the limit above, which is $x(t)$, has nonnegative entries now let's assume that $A_{ij} < 0$ for some $i \neq j$; we'll find trajectory with $x(0) \in \mathbf{R}^n_+$ but $x(t) \notin \mathbf{R}^n_+$ for some $t > 0$

let's take $x(0) = e_j$, so for small $h > 0$, we have $x(h) \approx e_j + hAe_j$

in particular, $x(h)_i \approx hA_{ij} < 0$ for small positive h, $i.e.$, $x(h) \notin \mathbf{R}^n_+$

this shows that if $A_{ij} < 0$ for some $i \neq j$, $\mathbf{R}^n_+$ isn't invariant

Conserved quantities

scalar valued function $\phi : \mathbf{R}^n \to \mathbf{R}$ is called *integral of the motion*, a *conserved quantity*, or *invariant* for $\dot{x} = f(x)$ if for every trajectory x, $\phi(x(t))$ is constant

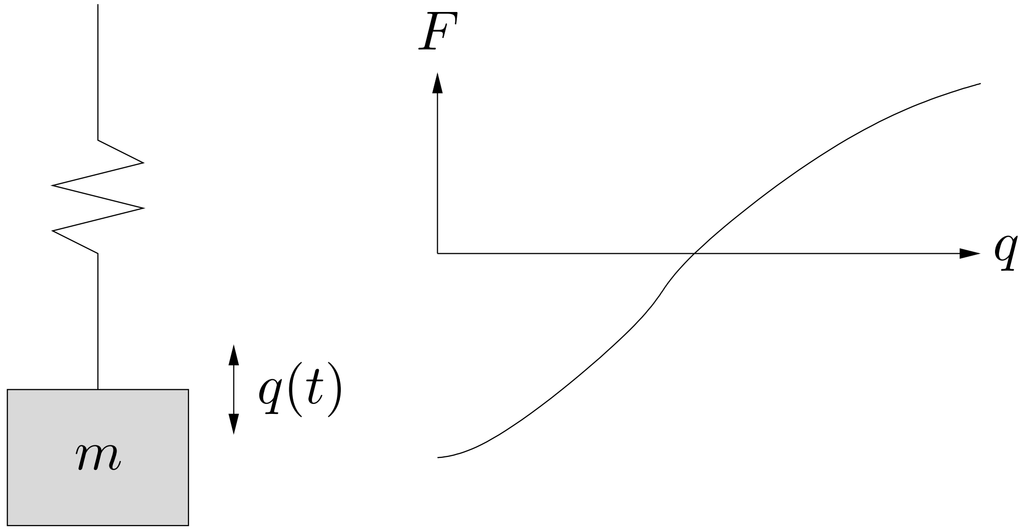
classical examples:

- total energy of a lossless mechanical system
- total angular momentum about an axis of an isolated system
- total fluid in a closed system

level set or *level surface* of ϕ, $\{z \in \mathbf{R}^n \mid \phi(z) = a\}$, are invariant sets

$e.g.$, trajectories of lossless mechanical system stay in surfaces of constant energy

Example: nonlinear lossless mechanical system



$m\ddot{q} = -F = -\phi(q)$, where $m > 0$ is mass, $q(t)$ is displacement, F is restoring force, ϕ is nonlinear spring characteristic with $\phi(0) = 0$

with $x = (q, \dot{q})$, we have

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} x_2 \\ -(1/m)\phi(x_1) \end{bmatrix}$$

potential energy stored in spring is

$$\psi(q) = \int_0^q \phi(u)\, du$$

total energy is kinetic plus potential: $E(x) = (m/2)\dot{q}^2 + \psi(q)$

E is a conserved quantity: if x is a trajectory, then

$$\begin{aligned} \frac{d}{dt}E(x(t)) &= (m/2)\frac{d}{dt}\dot{q}^2 + \frac{d}{dt}\psi(q) \\ &= m\dot{q}\ddot{q} + \phi(q)\dot{q} \\ &= m\dot{q}(-(1/m)\phi(q)) + \phi(q)\dot{q} \\ &= 0 \end{aligned}$$

$i.e.$, $E(x(t))$ is constant

Derivative of function along trajectory

we have function $\phi : \mathbf{R}^n \to \mathbf{R}$ and $\dot{x} = f(x)$

if x is trajectory of system, then

$$\frac{d}{dt}\phi(x(t)) = D\phi(x(t))\frac{dx}{dt} = \nabla\phi(x(t))^T f(x)$$

we define $\dot{\phi} : \mathbf{R}^n \to \mathbf{R}$ as

$$\dot{\phi}(z) = \nabla\phi(z)^T f(z)$$

intepretation: $\dot{\phi}(z)$ gives $\frac{d}{dt}\phi(x(t))$, if $x(t) = z$

$e.g.$, if $\dot{\phi}(z) > 0$, then $\phi(x(t))$ is increasing when $x(t)$ passes through z

if ϕ is conserved, then $\phi(x(t))$ is constant along any trajectory, so

$$\dot{\phi}(z) = \nabla \phi(z)^T f(x) = 0$$

for all z

this means the vector field $f(z)$ is everywhere orthogonal to $\nabla \phi$, which is normal to the level surface

Dissipated quantities

we say that $\phi : \mathbf{R}^n \to \mathbf{R}$ is a *dissipated quantity* for system $\dot{x} = f(x)$ if for all trajectories, $\phi(x(t))$ is (weakly) decreasing, *i.e.*, $\phi(x(\tau)) \leq \phi(x(t))$ for all $\tau \geq t$

classical examples:

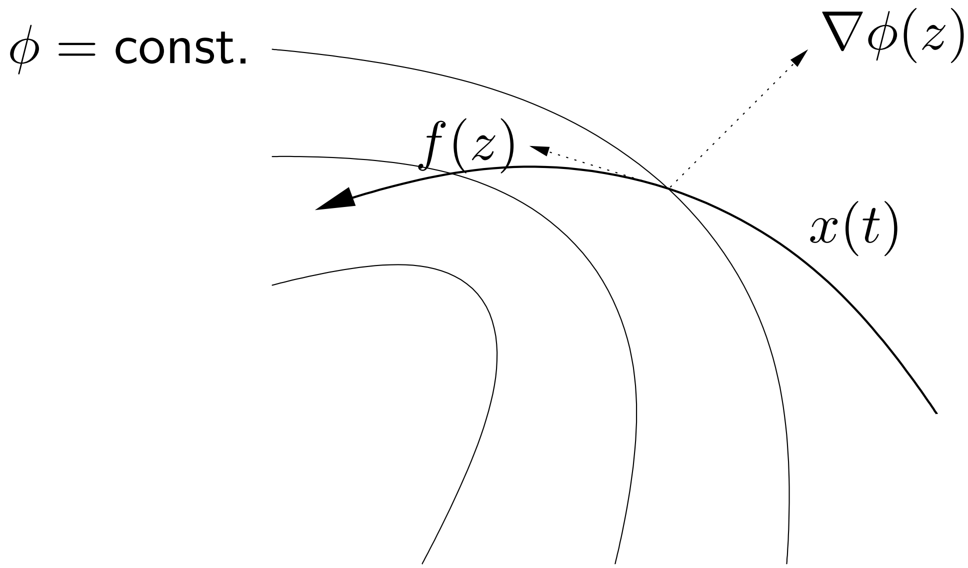
- total energy of a mechanical system with damping
- total fluid in a system that leaks

condition: $\dot{\phi}(z) \leq 0$ for all z, *i.e.*, $\nabla\phi(z)^T f(z) \leq 0$

$-\dot{\phi}$ is sometimes called the *dissipation function*

if ϕ is dissipated quantity, *sublevel sets* $\{z \mid \phi(z) \leq a\}$ are invariant

Geometric interpretation



- vector field points *into* sublevel sets
- $\nabla\phi(z)^T f(z) \leq 0$, $i.e.$, $\nabla\phi$ and f always make an obtuse angle
- trajectories can only "slip down" to lower values of ϕ

Example

linear mechanical system with damping: $M\ddot{q} + D\dot{q} + Kq = 0$

- $q(t) \in \mathbf{R}^n$ is displacement or configuration
- $M = M^T > 0$ is mass or inertia matrix
- $K = K^T > 0$ is stiffness matrix
- $D = D^T \geq 0$ is damping or loss matrix

we'll use state $x = (q, \dot{q})$, so

$$\dot{x} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x$$

consider total (potential plus kinetic) energy

$$E = \frac{1}{2}q^T K q + \frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}x^T \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} x$$

we have

$$\begin{aligned}
\dot{E}(z) &= \nabla E(z)^T f(z) \\
&= z^T \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} z \\
&= z^T \begin{bmatrix} 0 & K \\ -K & -D \end{bmatrix} z \\
&= -\dot{q}^T D \dot{q} \leq 0
\end{aligned}$$

makes sense: $\dfrac{d}{dt}$ (total stored energy) $= -$ (power dissipated)

Trajectory limit with dissipated quantity

suppose $\phi : \mathbf{R}^n \to \mathbf{R}$ is dissipated quantity for $\dot{x} = f(x)$

- $\phi(x(t)) \to \phi^*$ as $t \to \infty$, where $\phi^* \in \mathbf{R} \cup \{-\infty\}$
- if trajectory x is bounded and $\dot{\phi}$ is continuous, $x(t)$ converges to the *zero-dissipation set*:

$$x(t) \to \mathcal{D}_0 = \{z \mid \dot{\phi}(z) = 0\}$$

  $i.e.$, $\mathbf{dist}\,(x(t), \mathcal{D}_0) \to 0$, as $t \to \infty$ (more on this later)

Linear functions and linear dynamical systems

we consider linear system $\dot{x} = Ax$

when is a linear function $\phi(z) = c^T z$ conserved or dissipated?

$$\dot{\phi} = \nabla\phi(z)^T f(z) = c^T A z$$

$$\dot{\phi}(z) \leq 0 \text{ for all } z \iff \dot{\phi}(z) = 0 \text{ for all } z \iff A^T c = 0$$

$i.e.$, ϕ is dissipated if only if it is conserved, if and only if if $A^T c = 0$
(c is left eigenvector of A with eigenvalue 0)

Quadratic functions and linear dynamical systems

we consider linear system $\dot{x} = Ax$

when is a quadratic form $\phi(z) = z^T P z$ conserved or dissipated?

$$\dot{\phi}(z) = \nabla\phi(z)^T f(z) = 2z^T PAz = z^T(A^T P + PA)z$$

$i.e.$, $\dot{\phi}$ is also a quadratic form

- ϕ is conserved if and only if $A^T P + PA = 0$
  (which means A and $-A$ share at least $\mathbf{Rank}(P)$ eigenvalues)
- ϕ is dissipated if and only if $A^T P + PA \leq 0$

A criterion for invariance

suppose $\phi : \mathbf{R}^n \to \mathbf{R}$ satisfies $\phi(z) = 0 \implies \dot{\phi}(z) < 0$

then the set $C = \{z \mid \phi(z) \leq 0\}$ is invariant

idea: all trajectories on boundary of C cut *into* C, so none can leave

to show this, suppose trajectory x satisfies $x(t) \in C$, $x(s) \notin C$, $t \leq s$

consider (differentiable) function $g : \mathbf{R} \to \mathbf{R}$ given by $g(\tau) = \phi(x(\tau))$

g satisfies $g(t) \leq 0$, $g(s) > 0$

any such function must have at least one point $T \in [t, s]$ where $g(T) = 0$, $g'(T) \geq 0$ (for example, we can take $T = \min\{\tau \geq t \mid g(\tau) = 0\}$)

this means $\phi(x(T)) = 0$ and $\dot{\phi}(x(T)) \geq 0$, a contradiction

Discrete-time systems

we consider nonlinear time-invariant discrete-time system or recursion $x(t+1) = f(x(t))$

we say $C \subseteq \mathbf{R}^n$ is invariant (with respect to the system) if for every trajectory x,

$$x(t) \in C \implies x(\tau) \in C \text{ for all } \tau \geq t$$

$i.e.$, trajectories can enter, but cannot leave set C

equivalent to: $z \in C \implies f(z) \in C$

example: when is nonnegative orthant $\mathbf{R}^n_+$ invariant for $x(t+1) = Ax(t)$?

answer: $\Leftrightarrow A_{ij} \geq 0$ for $i,\ j = 1, \ldots, n$

Conserved and dissipated quantities

$\phi : \mathbf{R}^n \to \mathbf{R}$ is conserved under $x(t+1) = f(x(t))$ if $\phi(x(t))$ is constant, *i.e.*, $\phi(f(z)) = \phi(z)$ for all z

ϕ is a dissipated quantity if $\phi(x(t))$ is (weakly) decreasing, *i.e.*, $\phi(f(z)) \leq \phi(z)$ for all z

we define $\Delta\phi : \mathbf{R}^n \to \mathbf{R}$ by $\Delta\phi(z) = \phi(f(z)) - \phi(z)$

$\Delta\phi(z)$ gives change in ϕ, over one step, starting at z

ϕ is conserved if and only if $\Delta\phi(z) = 0$ for all z

ϕ is dissipated if and only if $\Delta\phi(z) \leq 0$ for all z

Quadratic functions and linear dynamical systems

we consider linear system $x(t+1) = Ax(t)$

when is a quadratic form $\phi(z) = z^T P z$ conserved or dissipated?

$$\Delta\phi(z) = (Az)^T P(Az) - z^T P z = z^T (A^T P A - P) z$$

$i.e.$, $\Delta\phi$ is also a quadratic form

- ϕ is conserved if and only if $A^T P A - P = 0$
  (which means A and A^{-1} share at least $\mathbf{Rank}(P)$ eigenvalues, if A invertible)
- ϕ is dissipated if and only if $A^T P A - P \leq 0$

Lecture 10
Basic Lyapunov theory

- stability
- positive definite functions
- global Lyapunov stability theorems
- Lasalle's theorem
- converse Lyapunov theorems
- finding Lyapunov functions

Some stability definitions

we consider nonlinear time-invariant system $\dot{x} = f(x)$, where $f : \mathbf{R}^n \to \mathbf{R}^n$

a point $x_e \in \mathbf{R}^n$ is an *equilibrium point* of the system if $f(x_e) = 0$

x_e is an equilibrium point $\Longleftrightarrow x(t) = x_e$ is a trajectory

suppose x_e is an equilibrium point

- system is *globally asymptotically stable* (G.A.S.) if for every trajectory $x(t)$, we have $x(t) \to x_e$ as $t \to \infty$
  (implies x_e is the unique equilibrium point)
- system is *locally asymptotically stable* (L.A.S.) near or at x_e if there is an $R > 0$ s.t. $\|x(0) - x_e\| \leq R \Longrightarrow x(t) \to x_e$ as $t \to \infty$

- often we change coordinates so that $x_e = 0$ ($i.e.$, we use $\tilde{x} = x - x_e$)
- a linear system $\dot{x} = Ax$ is G.A.S. (with $x_e = 0$) $\Leftrightarrow \Re\lambda_i(A) < 0$, $i = 1, \ldots, n$
- a linear system $\dot{x} = Ax$ is L.A.S. (near $x_e = 0$) $\Leftrightarrow \Re\lambda_i(A) < 0$, $i = 1, \ldots, n$
  (so for linear systems, L.A.S. $\Leftrightarrow$ G.A.S.)
- there are *many* other variants on stability ($e.g.$, stability, uniform stability, exponential stability, . . .)
- when f is nonlinear, establishing any kind of stability is usually very difficult

Energy and dissipation functions

consider nonlinear system $\dot{x} = f(x)$, and function $V : \mathbf{R}^n \to \mathbf{R}$

we define $\dot{V} : \mathbf{R}^n \to \mathbf{R}$ as $\dot{V}(z) = \nabla V(z)^T f(z)$

$\dot{V}(z)$ gives $\dfrac{d}{dt}V(x(t))$ when $z = x(t)$, $\dot{x} = f(x)$

we can think of V as *generalized energy function*, and $-\dot{V}$ as the associated *generalized dissipation function*

Positive definite functions

a function $V : \mathbf{R}^n \to \mathbf{R}$ is *positive definite* (PD) if

- $V(z) \geq 0$ for all z
- $V(z) = 0$ if and only if $z = 0$
- all sublevel sets of V are bounded

last condition equivalent to $V(z) \to \infty$ as $z \to \infty$

example: $V(z) = z^T P z$, with $P = P^T$, is PD if and only if $P > 0$

Lyapunov theory

Lyapunov theory is used to make conclusions about trajectories of a system $\dot{x} = f(x)$ ($e.g.$, G.A.S.) *without finding the trajectories* ($i.e.$, solving the differential equation)

a typical Lyapunov theorem has the form:

- **if** there exists a function $V : \mathbf{R}^n \to \mathbf{R}$ that satisfies some conditions on V and $\dot{V}$
- **then**, trajectories of system satisfy some property

if such a function V exists we call it a *Lyapunov function* (that proves the property holds for the trajectories)

Lyapunov function V can be thought of as *generalized energy function* for system

A Lyapunov boundedness theorem

suppose there is a function V that satisfies

- all sublevel sets of V are bounded
- $\dot{V}(z) \leq 0$ for all z

then, all trajectories are bounded, *i.e.*, for each trajectory x there is an R such that $\|x(t)\| \leq R$ for all $t \geq 0$

in this case, V is called a Lyapunov function (for the system) that proves the trajectories are bounded to prove it, we note that for any trajectory x

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \leq V(x(0))$$

so the whole trajectory lies in $\{z \mid V(z) \leq V(x(0))\}$, which is bounded

also shows: every sublevel set $\{z \mid V(z) \leq a\}$ is invariant

A Lyapunov global asymptotic stability theorem

Suppose there is a function V such that

- V is positive definite
- $\dot{V}(z) < 0$ for all $z \neq 0$, $\dot{V}(0) = 0$

then, every trajectory of $\dot{x} = f(x)$ converges to zero as $t \to \infty$
($i.e.$, the system is globally asymptotically stable)

intepretation:

- V is positive definite generalized energy function
- energy is always dissipated, except at 0

Proof

Suppose trajectory $x(t)$ does not converge to zero.

$V(x(t))$ is decreasing and nonnegative, so it converges to, say, ϵ as $t \to \infty$.

Since $x(t)$ doesn't converge to 0, we must have $\epsilon > 0$, so for all t, $\epsilon \leq V(x(t)) \leq V(x(0))$.

$C = \{z \mid \epsilon \leq V(z) \leq V(x(0))\}$ is closed and bounded, hence compact. So $\dot{V}$ (assumed continuous) attains its supremum on C, *i.e.*, $\sup_{z \in C} \dot{V} = -a < 0$. Since $\dot{V}(x(t)) \leq -a$ for all t, we have

$$V(x(T)) = V(x(0)) + \int_0^T \dot{V}(z)\, dz \leq V(x(0)) - aT$$

which for $T > V(x(0))/a$ implies $V(x(0)) < 0$, a contradiction.

So every trajectory $x(t)$ converges to 0, *i.e.*, $\dot{x} = f(x)$ is G.A.S.

A Lyapunov exponential stability theorem

suppose there is a function V and constant $\alpha > 0$ such that

- V is positive definite
- $\dot{V}(z) \leq -\alpha V(z)$ for all z

then, there is an M such that every trajectory of $\dot{x} = f(x)$ satisfies
$\|x(t)\| \leq Me^{-\alpha t/2}\|x(0)\|$
(this is called *global exponential stability* (G.E.S.))

idea: $\dot{V} \leq -\alpha V$ gives guaranteed minimum dissipation rate, proportional to energy

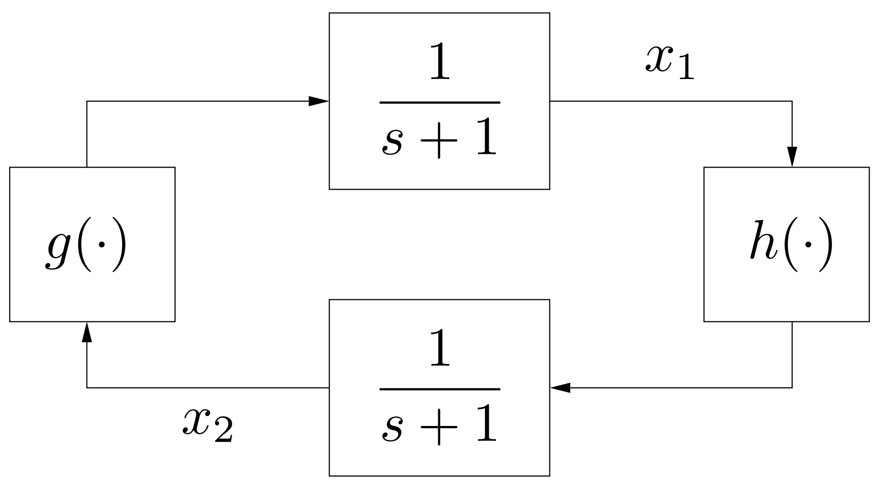
Example

consider system

$$\dot{x}_1 = -x_1 + g(x_2), \qquad \dot{x}_2 = -x_2 + h(x_1)$$

where $|g(u)| \leq |u|/2$, $|h(u)| \leq |u|/2$

two first order systems with nonlinear cross-coupling



let's use Lyapunov theorem to show it's globally asymptotically stable

we use $V = (x_1^2 + x_2^2)/2$

required properties of V are clear ($V \geq 0$, etc.)

let's bound $\dot{V}$:

$$\begin{aligned}
\dot{V} &= x_1\dot{x}_1 + x_2\dot{x}_2 \\
&= -x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1) \\
&\leq -x_1^2 - x_2^2 + |x_1 x_2| \\
&\leq -(1/2)(x_1^2 + x_2^2) \\
&= -V
\end{aligned}$$

where we use $|x_1 x_2| \leq (1/2)(x_1^2 + x_2^2)$ (derived from $(|x_1| - |x_2|)^2 \geq 0$)

we conclude system is G.A.S. (in fact, G.E.S.)
without knowing the trajectories

Lasalle's theorem

Lasalle's theorem (1960) allows us to conclude G.A.S. of a system with only $\dot{V} \leq 0$, along with an observability type condition

we consider $\dot{x} = f(x)$

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- V is positive definite
- $\dot{V}(z) \leq 0$
- the only solution of $\dot{w} = f(w)$, $\dot{V}(w) = 0$ is $w(t) = 0$ for all t

then, the system $\dot{x} = f(x)$ is G.A.S.

- last condition means no nonzero trajectory can hide in the “zero dissipation” set
- unlike most other Lyapunov theorems, which extend to time-varying systems, Lasalle’s theorem *requires* time-invariance

A Lyapunov instability theorem

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $\dot{V}(z) \leq 0$ for all z (or just whenever $V(z) \leq 0$)
- there is w such that $V(w) < V(0)$

then, the trajectory of $\dot{x} = f(x)$ with $x(0) = w$ does not converge to zero (and therefore, the system is not G.A.S.)

to show it, we note that $V(x(t)) \leq V(x(0)) = V(w) < V(0)$ for all $t \geq 0$

but if $x(t) \to 0$, then $V(x(t)) \to V(0)$; so we cannot have $x(t) \to 0$

A Lyapunov divergence theorem

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $\dot{V}(z) < 0$ whenever $V(z) < 0$
- there is w such that $V(w) < 0$

then, the trajectory of $\dot{x} = f(x)$ with $x(0) = w$ is unbounded, *i.e.*,

$$\sup_{t \geq 0} \|x(t)\| = \infty$$

(this is not quite the same as $\lim_{t \to \infty} \|x(t)\| = \infty$)

Proof of Lyapunov divergence theorem

let $\dot{x} = f(x)$, $x(0) = w$. let's first show that $V(x(t)) \leq V(w)$ for all $t \geq 0$.

if not, let T denote the smallest positive time for which $V(x(T)) = V(w)$. then over $[0, T]$, we have $V(x(t)) \leq V(w) < 0$, so $\dot{V}(x(t)) < 0$, and so

$$\int_0^T \dot{V}(x(t))\, dt < 0$$

the lefthand side is also equal to

$$\int_0^T \dot{V}(x(t))\, dt = V(x(T)) - V(x(0)) = 0$$

so we have a contradiction.

it follows that $V(x(t)) \leq V(x(0))$ for all t, and therefore $\dot{V}(x(t)) < 0$ for all t.

now suppose that $\|x(t)\| \leq R$, *i.e.*, the trajectory is bounded.

$\{z \mid V(z) \leq V(x(0)),\ \|z\| \leq R\}$ is compact, so there is a $\beta > 0$ such that $\dot{V}(z) \leq -\beta$ whenever $V(z) \leq V(x(0))$ and $\|z\| \leq R$.

we conclude $V(x(t)) \leq V(x(0)) - \beta t$ for all $t \geq 0$, so $V(x(t)) \rightarrow -\infty$, a contradiction.

Converse Lyapunov theorems

a typical *converse Lyapunov theorem* has the form

- **if** the trajectories of system satisfy some property
- **then** there exists a Lyapunov function that proves it

a sharper converse Lyapunov theorem is more specific about the form of the Lyapunov function

example: if the linear system $\dot{x} = Ax$ is G.A.S., then there is a quadratic Lyapunov function that proves it (we'll prove this later)

A converse Lyapunov G.E.S. theorem

suppose there is $\beta > 0$ and M such that each trajectory of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq M e^{-\beta t} \|x(0)\| \text{ for all } t \geq 0$$

(called *global exponential stability*, and is stronger than G.A.S.)

then, there is a Lyapunov function that proves the system is exponentially stable, $i.e.$, there is a function $V : \mathbf{R}^n \to \mathbf{R}$ and constant $\alpha > 0$ s.t.

- V is positive definite
- $\dot{V}(z) \leq -\alpha V(z)$ for all z

Proof of converse G.A.S. Lyapunov theorem

suppose the hypotheses hold, and define

$$V(z) = \int_0^\infty \|x(t)\|^2 \, dt$$

where $x(0) = z$, $\dot{x} = f(x)$

since $\|x(t)\| \leq Me^{-\beta t}\|z\|$, we have

$$V(z) = \int_0^\infty \|x(t)\|^2 \, dt \leq \int_0^\infty M^2 e^{-2\beta t}\|z\|^2 \, dt = \frac{M^2}{2\beta}\|z\|^2$$

(which shows integral is finite)

let's find $\dot{V}(z) = \left.\dfrac{d}{dt}\right|_{t=0} V(x(t))$, where $x(t)$ is trajectory with $x(0) = z$

$$\begin{aligned}
\dot{V}(z) &= \lim_{t\to 0}(1/t)\left(V(x(t)) - V(x(0))\right) \\
&= \lim_{t\to 0}(1/t)\left(\int_t^\infty \|x(\tau)\|^2\, d\tau - \int_0^\infty \|x(\tau)\|^2\, d\tau\right) \\
&= \lim_{t\to 0}(-1/t)\int_0^t \|x(\tau)\|^2\, d\tau \\
&= -\|z\|^2
\end{aligned}$$

now let's verify properties of V

$V(z) \geq 0$ and $V(z) = 0 \;\Leftrightarrow\; z = 0$ are clear

finally, we have $\dot{V}(z) = -z^T z \leq -\alpha V(z)$, with $\alpha = 2\beta/M^2$

Finding Lyapunov functions

- there are many different types of Lyapunov theorems
- the key in all cases is to *find* a Lyapunov function and verify that it has the required properties
- there are several approaches to finding Lyapunov functions and verifying the properties

one common approach:

- decide form of Lyapunov function ($e.g.$, quadratic), parametrized by some parameters (called a *Lyapunov function candidate*)
- try to find values of parameters so that the required hypotheses hold

Other sources of Lyapunov functions

- value function of a related optimal control problem
- linear-quadratic Lyapunov theory (next lecture)
- computational methods
- converse Lyapunov theorems
- graphical methods (really!)

(as you might guess, these are all somewhat related)

Lecture 11
Linear quadratic Lyapunov theory

- the Lyapunov equation
- Lyapunov stability conditions
- the Lyapunov operator and integral
- evaluating quadratic integrals
- analysis of ARE
- discrete-time results
- linearization theorem

The Lyapunov equation

the *Lyapunov equation* is

$$A^T P + PA + Q = 0$$

where $A,\ P,\ Q \in \mathbf{R}^{n\times n}$, and $P,\ Q$ are symmetric

interpretation: for linear system $\dot{x} = Ax$, if $V(z) = z^T P z$, then

$$\dot{V}(z) = (Az)^T P z + z^T P(Az) = -z^T Q z$$

$i.e.$, if $z^T P z$ is the (generalized)*energy*, then $z^T Q z$ is the associated (generalized) *dissipation*

linear-quadratic Lyapunov theory: *linear* dynamics, *quadratic* Lyapunov function we consider system $\dot{x} = Ax$, with $\lambda_1, \ldots, \lambda_n$ the eigenvalues of A

if $P > 0$, then

- the sublevel sets are ellipsoids (and bounded)
- $V(z) = z^T P z = 0 \;\Leftrightarrow\; z = 0$

boundedness condition: if $P > 0$, $Q \geq 0$ then

- all trajectories of $\dot{x} = Ax$ are bounded
(this means $\Re\lambda_i \leq 0$, and if $\Re\lambda_i = 0$, then λ_i corresponds to a Jordan block of size one)
- the ellipsoids $\{z \mid z^T P z \leq a\}$ are invariant

Stability condition

if $P > 0$, $Q > 0$ then the system $\dot{x} = Ax$ is (globally asymptotically) stable, *i.e.*, $\Re\lambda_i < 0$

to see this, note that

$$\dot{V}(z) = -z^T Q z \leq -\lambda_{\min}(Q) z^T z \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} z^T P z = -\alpha V(z)$$

where $\alpha = \lambda_{\min}(Q)/\lambda_{\max}(P) > 0$

An extension based on observability

(Lasalle's theorem for linear dynamics, quadratic function)

if $P > 0$, $Q \geq 0$, and (Q, A) observable, then the system $\dot{x} = Ax$ is (globally asymptotically) stable

to see this, we first note that all eigenvalues satisfy $\Re\lambda_i \leq 0$

now suppose that $v \neq 0$, $Av = \lambda v$, $\Re\lambda = 0$

then $A\bar{v} = \bar{\lambda}\bar{v} = -\lambda\bar{v}$, so

$$\left\| Q^{1/2} v \right\|^2 = v^* Q v = -v^* \left(A^T P + PA \right) v = \lambda v^* P v - \lambda v^* P v = 0$$

which implies $Q^{1/2} v = 0$, so $Qv = 0$, contradicting observability (by PBH)

interpretation: observability condition means no trajectory can stay in the "zero dissipation" set $\{z \mid z^T Q z = 0\}$

An instability condition

if $Q \geq 0$ and $P \not\geq 0$, then A is not stable

to see this, note that $\dot{V} \leq 0$, so $V(x(t)) \leq V(x(0))$

since $P \not\geq 0$, there is a w with $V(w) < 0$; trajectory starting at w does not converge to zero

in this case, the sublevel sets $\{z \mid V(z) \leq 0\}$ (which are unbounded) are invariant

The Lyapunov operator

the *Lyapunov operator* is given by

$$\mathcal{L}(P) = A^T P + PA$$

special case of Sylvester operator

$\mathcal{L}$ is nonsingular if and only if A and $-A$ share no common eigenvalues, $i.e.$, A does not have pair of eigenvalues which are negatives of each other

- if A is stable, Lyapunov operator is nonsingular
- if A has imaginary (nonzero, $j\omega$-axis) eigenvalue, then Lyapunov operator is singular

thus if A is stable, for any Q there is exactly one solution P of Lyapunov equation $A^T P + PA + Q = 0$

Solving the Lyapunov equation

$$A^T P + PA + Q = 0$$

we are given A and Q and want to find P

if Lyapunov equation is solved as a set of n^2 equations in n^2 variables, cost is $O(n^6)$ operations

fast methods, that exploit the special structure of the linear equations, can solve Lyapunov equation with cost $O(n^3)$

based on first reducing A to Schur or upper Hessenberg form

The Lyapunov integral

if A is stable there is an explicit formula for solution of Lyapunov equation:

$$P = \int_0^\infty e^{tA^T} Q e^{tA}\, dt$$

to see this, we note that

$$\begin{aligned}
A^T P + PA &= \int_0^\infty \left(A^T e^{tA^T} Q e^{tA} + e^{tA^T} Q e^{tA} A \right) \, dt \\
&= \int_0^\infty \left(\frac{d}{dt} e^{tA^T} Q e^{tA} \right) \, dt \\
&= e^{tA^T} Q e^{tA} \Big|_0^\infty \\
&= -Q
\end{aligned}$$

Interpretation as cost-to-go

if A is stable, and P is (unique) solution of $A^TP + PA + Q = 0$, then

$$\begin{aligned} V(z) &= z^TPz \\ &= z^T\left(\int_0^\infty e^{tA^T}Qe^{tA}\,dt\right)z \\ &= \int_0^\infty x(t)^TQx(t)\,dt \end{aligned}$$

where $\dot{x} = Ax$, $x(0) = z$

thus $V(z)$ is cost-to-go from point z (with no input) and integral quadratic cost function with matrix Q

if A is stable and $Q > 0$, then for each t, $e^{tA^T} Q e^{tA} > 0$, so

$$P = \int_0^\infty e^{tA^T} Q e^{tA}\, dt > 0$$

meaning: if A is stable,

- we can choose *any* positive definite quadratic form $z^T Q z$ as the dissipation, $i.e.$, $-\dot{V} = z^T Q z$
- then solve a set of linear equations to find the (unique) quadratic form $V(z) = z^T P z$
- V will be positive definite, so it is a Lyapunov function that proves A is stable

in particular: *a linear system is stable if and only if there is a quadratic Lyapunov function that proves it*

generalization: if A stable, $Q \geq 0$, and (Q, A) observable, then $P > 0$

to see this, the Lyapunov integral shows $P \geq 0$

if $Pz = 0$, then

$$0 = z^T P z = z^T \left(\int_0^\infty e^{tA^T} Q e^{tA} \, dt \right) z = \int_0^\infty \left\| Q^{1/2} e^{tA} z \right\|^2 \, dt$$

so we conclude $Q^{1/2} e^{tA} z = 0$ for all $t \geq 0$

this implies that $Qz = 0$, $QAz = 0$, . . . , $QA^{n-1}z = 0$, contradicting (Q, A) observable

Monotonicity of Lyapunov operator inverse

suppose $A^TP_i + P_iA + Q_i = 0$, $i = 1,\ 2$

if $Q_1 \geq Q_2$, then for all t, $e^{tA^T}Q_1e^{tA} \geq e^{tA^T}Q_2e^{tA}$

if A is stable, we have

$$P_1 = \int_0^\infty e^{tA^T}Q_1e^{tA}\,dt \geq \int_0^\infty e^{tA^T}Q_2e^{tA}\,dt = P_2$$

in other words: if A is stable then

$$Q_1 \geq Q_2 \implies \mathcal{L}^{-1}(Q_1) \geq \mathcal{L}^{-1}(Q_2)$$

i.e., inverse Lyapunov operator is monotonic, or preserves matrix inequality, when A is stable

(question: is $\mathcal{L}$ monotonic?)

Evaluating quadratic integrals

suppose $\dot{x} = Ax$ is stable, and define

$$J = \int_0^\infty x(t)^T Q x(t)\, dt$$

to find J, we solve Lyapunov equation $A^T P + PA + Q = 0$ for P

then, $J = x(0)^T P x(0)$

in other words: we can evaluate quadratic integral exactly, by solving a set of linear equations, without even computing a matrix exponential

Controllability and observability Grammians

for A stable, the controllability Grammian of (A, B) is defined as

$$W_c = \int_0^\infty e^{tA} B B^T e^{tA^T} \, dt$$

and the observability Grammian of (C, A) is

$$W_o = \int_0^\infty e^{tA^T} C^T C e^{tA} \, dt$$

these Grammians can be computed by solving the Lyapunov equations

$$AW_c + W_c A^T + BB^T = 0, \qquad A^T W_o + W_o A + C^T C = 0$$

we always have $W_c \geq 0$, $W_o \geq 0$;
$W_c > 0$ if and only if (A, B) is controllable, and
$W_o > 0$ if and only if (C, A) is observable

Evaluating a state feedback gain

consider

$$\dot{x} = Ax + Bu, \qquad y = Cx, \qquad u = Kx, \qquad x(0) = x_0$$

with closed-loop system $\dot{x} = (A + BK)x$ stable

to evaluate the quadratic integral performance measures

$$J_u = \int_0^\infty u(t)^T u(t)\, dt, \qquad J_y = \int_0^\infty y(t)^T y(t)\, dt$$

we solve Lyapunov equations

$$\begin{aligned} (A + BK)^T P_u + P_u(A + BK) + K^T K &= 0 \\ (A + BK)^T P_y + P_y(A + BK) + C^T C &= 0 \end{aligned}$$

then we have $J_u = x_0^T P_u x_0$, $J_y = x_0^T P_y x_0$

Lyapunov analysis of ARE

write ARE (with $Q \geq 0$, $R > 0$)

$$A^T P + PA + Q - PBR^{-1}B^T P = 0$$

as

$$(A + BK)^T P + P(A + BK) + (Q + K^T RK) = 0$$

with $K = -R^{-1}B^T P$

we conclude: if $A + BK$ stable, then $P \geq 0$ (since $Q + K^T RK \geq 0$)

$i.e.$, any stabilizing solution of ARE is PSD

if also (Q, A) is observable, then we conclude $P > 0$

to see this, we need to show that $(Q + K^T RK, A + BK)$ is observable

if not, there is $v \neq 0$ s.t.

$$(A + BK)v = \lambda v, \qquad (Q + K^T RK)v = 0$$

which implies

$$v^*(Q + K^T RK)v = v^*Qv + v^*K^T RKv = \|Q^{1/2}v\|^2 + \|R^{1/2}Kv\|^2 = 0$$

so $Qv = 0$, $Kv = 0$

$$(A + BK)v = Av = \lambda v, \qquad Qv = 0$$

which contradicts (Q, A) observable

the same argument shows that if $P > 0$ and (Q, A) is observable, then $A + BK$ is stable

Monotonic norm convergence

suppose that $A + A^T < 0$, $i.e.$, (symmetric part of) A is negative definite

can express as $A^T P + PA + Q = 0$, with $P = I$, $Q > 0$

meaning: $x^T P x = \|x\|^2$ decreases along every nonzero trajectory, $i.e.$,

- $\|x(t)\|$ is always *decreasing monotonically* to 0
- $x(t)$ is always moving towards origin

this implies A is stable, but the converse is false: for a stable system, we need not have $A + A^T < 0$

(for a stable system with $A + A^T \not< 0$, $\|x(t)\|$ converges to zero, but not monotonically) for a stable system we can always change coordinates so we have monotonic norm convergence

let P denote the solution of $A^TP + PA + I = 0$

take $T = P^{-1/2}$

in new coordinates A becomes $\tilde{A} = T^{-1}AT$,

$$\begin{aligned}\tilde{A} + \tilde{A}^T &= P^{1/2}AP^{-1/2} + P^{-1/2}A^TP^{1/2} \\ &= P^{-1/2}\left(PA + A^TP\right)P^{-1/2} \\ &= -P^{-1} < 0\end{aligned}$$

in new coordinates, convergence is *obvious* because $\|x(t)\|$ is always decreasing

Discrete-time results

all linear quadratic Lyapunov results have discrete-time counterparts

the *discrete-time* Lyapunov equation is

$$A^T PA - P + Q = 0$$

meaning: if $x(t+1) = Ax(t)$ and $V(z) = z^T Pz$, then $\Delta V(z) = -z^T Qz$

- if $P > 0$ and $Q > 0$, then A is (discrete-time) stable ($i.e.$, $|\lambda_i| < 1$)
- if $P > 0$ and $Q \geq 0$, then all trajectories are bounded ($i.e.$, $|\lambda_i| \leq 1$; $|\lambda_i| = 1$ only for 1×1 Jordan blocks)
- if $P > 0$, $Q \geq 0$, and (Q, A) observable, then A is stable
- if $P \not> 0$ and $Q \geq 0$, then A is not stable

Discrete-time Lyapunov operator

the discrete-time Lyapunov operator is given by $\mathcal{L}(P) = A^T PA - P$

$\mathcal{L}$ is nonsingular if and only if, for all $i,\ j$, $\lambda_i \lambda_j \neq 1$
(roughly speaking, if and only if A and A^{-1} share no eigenvalues)

if A is stable, then $\mathcal{L}$ is nonsingular; in fact

$$P = \sum_{t=0}^{\infty} (A^T)^t Q A^t$$

is the unique solution of Lyapunov equation $A^T PA - P + Q = 0$

the discrete-time Lyapunov equation can be solved quickly ($i.e.$, $O(n^3)$)
and can be used to evaluate infinite sums of quadratic functions, etc.

Converse theorems

suppose $x(t+1) = Ax(t)$ is stable, $A^TPA - P + Q = 0$

- if $Q > 0$ then $P > 0$
- if $Q \geq 0$ and (Q, A) observable, then $P > 0$

in particular, a discrete-time linear system is stable if and only if there is a quadratic Lyapunov function that proves it

Monotonic norm convergence

suppose $A^T PA - P + Q = 0$, with $P = I$ and $Q > 0$

this means $A^T A < I$, *i.e.*, $\|A\| < 1$

meaning: $\|x(t)\|$ decreases on every nonzero trajectory; indeed,
$\|x(t+1)\| \leq \|A\|\|x(t)\| < \|x(t)\|$

when $\|A\| < 1$,

- stability is obvious, since $\|x(t)\| \leq \|A\|^t \|x(0)\|$
- system is called *contractive* since norm is reduced at each step

the converse is false: system can be stable without $\|A\| < 1$

now suppose A is stable, and let P satisfy $A^T PA - P + I = 0$

take $T = P^{-1/2}$

in new coordinates A becomes $\tilde{A} = T^{-1}AT$, so

$$\begin{aligned} \tilde{A}^T \tilde{A} &= P^{-1/2} A^T PAP^{-1/2} \\ &= P^{-1/2}(P - I)P^{-1/2} \\ &= I - P^{-1} < I \end{aligned}$$

$i.e.$, $\|\tilde{A}\| < 1$

so for a stable system, we can change coordinates so the system is contractive

Lyapunov's linearization theorem

we consider nonlinear time-invariant system $\dot{x} = f(x)$, where $f : \mathbf{R}^n \to \mathbf{R}^n$

suppose x_e is an equilibrium point, *i.e.*, $f(x_e) = 0$, and let $A = Df(x_e) \in \mathbf{R}^{n \times n}$

the linearized system, near x_e, is $\dot{\delta x} = A\delta x$

linearization theorem:

- if the linearized system is stable, *i.e.*, $\Re\lambda_i(A) < 0$ for $i = 1, \ldots, n$, then the nonlinear system is locally asymptotically stable
- if for some i, $\Re\lambda_i(A) > 0$, then the nonlinear system is not locally asymptotically stable

stability of the linearized system determines the local stability of the nonlinear system, *except* when all eigenvalues are in the closed left halfplane, and at least one is on the imaginary axis

examples like $\dot{x} = x^3$ (which is not LAS) and $\dot{x} = -x^3$ (which is LAS) show the theorem cannot, in general, be tightened

examples:

eigenvalues of $Df(x_e)$	conclusion about $\dot{x} = f(x)$
$-3, \ -0.1 \pm 4j, \ -0.2 \pm j$	LAS near x_e
$-3, \ -0.1 \pm 4j, \ 0.2 \pm j$	not LAS near x_e
$-3, \ -0.1 \pm 4j, \ \pm j$	no conclusion

Proof of linearization theorem

let's assume $x_e = 0$, and express the nonlinear differential equation as

$$\dot{x} = Ax + g(x)$$

where $\|g(x)\| \leq K\|x\|^2$

suppose that A is stable, and let P be unique solution of Lyapunov equation

$$A^T P + PA + I = 0$$

the Lyapunov function $V(z) = z^T P z$ proves stability of the linearized system; we'll use it to prove local asymptotic stability of the nonlinear system

$$\begin{aligned}
\dot{V}(z) &= 2z^T P(Az + g(z)) \\
&= z^T(A^T P + PA)z + 2z^T Pg(z) \\
&= -z^T z + 2z^T Pg(z) \\
&\leq -\|z\|^2 + 2\|z\|\|P\|\|g(z)\| \\
&\leq -\|z\|^2 + 2K\|P\|\|z\|^3 \\
&= -\|z\|^2(1 - 2K\|P\|\|z\|)
\end{aligned}$$

so for $\|z\| \leq 1/(4K\|P\|)$,

$$\dot{V}(z) \leq -\frac{1}{2}\|z\|^2 \leq -\frac{1}{2\lambda_{\max}(P)} z^T P z = -\frac{1}{2\|P\|} z^T P z$$

finally, using

$$\|z\|^2 \leq \frac{1}{\lambda_{\min}(P)} z^T P z$$

we have

$$V(z) \leq \frac{\lambda_{\min}(P)}{16K^2\|P\|^2} \implies \|z\| \leq \frac{1}{4K\|P\|} \implies \dot{V}(z) \leq -\frac{1}{2\|P\|} V(z)$$

and we're done

comments:

- proof actually constructs an ellipsoid inside basin of attraction of $x_e = 0$, and a bound on exponential rate of convergence
- choice of $Q = I$ was arbitrary; can get better estimates using other Qs, better bounds on g, tighter bounding arguments . . .

Integral quadratic performance

consider $\dot{x} = f(x)$, $x(0) = x_0$

we are interested in the integral quadratic performance measure

$$J(x_0) = \int_0^\infty x(t)^T Q x(t)\, dt$$

for any fixed x_0 we can find this (approximately) by simulation and numerical integration

(we'll assume the integral exists; we do not require $Q \geq 0$)

Lyapunov bounds on integral quadratic performance

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $V(z) \geq 0$ for all z
- $\dot{V}(z) \leq -z^T Q z$ for all z

then we have $J(x_0) \leq V(x_0)$, *i.e.*, the Lyapunov function V serves as an upper bound on the integral quadratic cost

(since Q need not be PSD, we might not have $\dot{V} \leq 0$; so we cannot conclude that trajectories are bounded) to show this, we note that

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}(x(t))\, dt \le -\int_0^T x(t)^T Q x(t)\, dt$$

and so

$$\int_0^T x(t)^T Q x(t)\, dt \le V(x(0)) - V(x(T)) \le V(x(0))$$

since this holds for arbitrary T, we conclude

$$\int_0^\infty x(t)^T Q x(t)\, dt \le V(x(0))$$

Integral quadratic performance for linear systems

for a stable linear system, with $Q \geq 0$, the Lyapunov bound is sharp, $i.e.$, there exists a V such that

- $V(z) \geq 0$ for all z
- $\dot{V}(z) \leq -z^T Q z$ for all z

and for which $V(x_0) = J(x_0)$ for all x_0

(take $V(z) = z^T P z$, where P is solution of $A^T P + PA + Q = 0$)

Lecture 12
Lyapunov theory with inputs and outputs

- systems with inputs and outputs
- reachability bounding
- bounds on RMS gain
- bounded-real lemma
- feedback synthesis via control-Lyapunov functions

Systems with inputs

we now consider systems with inputs, $i.e.$, $\dot{x} = f(x, u)$, where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$

if $x,\ u$ is state-input trajectory and $V : \mathbf{R}^n \to \mathbf{R}$, then

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t))^T \dot{x}(t) = \nabla V(x(t))^T f(x(t), u(t))$$

so we define $\dot{V} : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ as

$$\dot{V}(z, w) = \nabla V(z)^T f(z, w)$$

($i.e.$, $\dot{V}$ depends on the state and input)

Reachable set with admissable inputs

consider $\dot{x} = f(x, u)$, $x(0) = 0$, and $u(t) \in \mathcal{U}$ for all t

$\mathcal{U} \subseteq \mathbf{R}^m$ is called the set of *admissable inputs*

we define the *reachable set* as

$$\mathcal{R} = \{x(T) \mid \dot{x} = f(x, u),\ x(0) = 0,\ u(t) \in \mathcal{U},\ T > 0\}$$

$i.e.$, the set of points that can be hit by a trajectory with some admissiable input

applications:

- if u is a control input that we can manipulate, $\mathcal{R}$ shows the places we can hit (so big $\mathcal{R}$ is good)
- if u is a disturbance, noise, or antagonistic signal (beyond our control), $\mathcal{R}$ shows the worst-case effect on x (so big $\mathcal{R}$ is bad)

Lyapunov bound on reachable set

Lyapunov arguments can be used to bound reachable sets of nonlinear or time-varying systems

suppose there is a $V : \mathbf{R}^n \to \mathbf{R}$ and $a > 0$ such that

$$\dot{V}(z, w) \le -a \text{ whenever } V(z) = b \text{ and } w \in \mathcal{U}$$

and define $C = \{z \mid V(z) \le b\}$

then, if $\dot{x} = f(x, u)$, $x(0) \in C$, and $u(t) \in \mathcal{U}$ for $0 \le t \le T$, we have $x(T) \in C$

$i.e.$, every trajectory that starts in $C = \{z \mid V(z) \le b\}$ stays there, for any admissable u

in particular, if $0 \in C$, we conclude $\mathcal{R} \subseteq C$

idea: on the boundary of C, every trajectory cuts *into* C, for all admissable values of u

proof: suppose $\dot{x} = f(x,u)$, $x(0) \in C$, and $u(t) \in \mathcal{U}$ for $0 \leq t \leq T$, and V satisfies hypotheses

suppose that $x(T) \notin C$

consider scalar function $g(t) = V(x(t))$

$g(0) \leq b$ and $g(T) > b$, so there is a $t_0 \in [0,T]$ with $g(t_0) = b$, $g'(t_0) \geq 0$

but

$$g'(t_0) = \frac{d}{dt} V(x(t)) = \dot{V}(x(t), u(t)) \leq -a < 0$$

by the hypothesis, so we have a contradiction

Reachable set with integral quadratic bounds

we consider $\dot{x} = f(x,u)$, $x(0) = 0$, with an integral constraint on the input:

$$\int_0^\infty u(t)^T u(t)\, dt \le a$$

the reachable set with this integral quadratic bound is

$$\mathcal{R}_a = \left\{ x(T) \;\middle|\; \dot{x} = f(x,u),\ x(0) = x_0,\ \int_0^T u(t)^T u(t)\, dt \le a \right\}$$

$i.e.$, the set of points that can be hit using at most a energy

Example

consider stable linear system $\dot{x} = Ax + Bu$

minimum energy ($i.e.$, integral of $u^T u$) to hit point z is $z^T W_c^{-1} z$, where W_c is controllability Grammian

reachable set with integral quadratic bound is (open) ellipsoid

$$\mathcal{R}_a = \{z \mid z^T W_c^{-1} z < a\}$$

Lyapunov bound on reachable set with integral constraint

suppose there is a $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $V(z) \geq 0$ for all z, $V(0) = 0$
- $\dot{V}(z, w) \leq w^T w$ for all $z,\ w$

then $\mathcal{R}_a \subseteq \{z \mid V(z) \leq a\}$

proof:

$$V(x(T)) - V(x(0)) = \int_0^T \dot{V}(x(t), u(t))\ dt \leq \int_0^T u(t)^T u(t)\ dt \leq a$$

so, using $V(x(0)) = V(0) = 0$, $V(x(T)) \leq a$

interpretation:

- V is (generalized) internally stored energy in system
- $u(t)^T u(t)$ is power supplied to system by input
- $\dot{V} \leq u^T u$ means stored energy increases by no more than power input
- $V(0) = 0$ means system starts in zero energy state
- conclusion is: if energy $\leq a$ applied, can only get to states with stored energy $\leq a$

Stable linear system

consider stable linear system $\dot{x} = Ax + Bu$

we'll show Lyapunov bound is tight in this case, with $V(z) = z^T W_c^{-1} z$

multiply $AW_c + W_c A^T + BB^T = 0$ on left & right by W_c^{-1} to get

$$W_c^{-1}A + A^T W_c^{-1} + W_c^{-1}BB^T W_c^{-1} = 0$$

now we can find and bound $\dot{V}$:

$$\begin{aligned}
\dot{V}(z,w) &= 2z^T W_c^{-1}(Az + Bw) \\
&= z^T \left(W_c^{-1}A + A^T W_c^{-1}\right) z + 2z^T W_c^{-1} Bw \\
&= -z^T W_c^{-1} BB^T W_c^{-1} z + 2z^T W_c^{-1} Bw \\
&= -\|B^T W_c^{-1} z - w\|^2 + w^T w \\
&\leq w^T w
\end{aligned}$$

for $V(z) = z^T W_c^{-1} z$, Lyapunov bound is

$$\mathcal{R}_a \subseteq \{z \mid z^T W_c^{-1} z \leq a\}$$

righthand set is closure of lefthand set, so bound is tight

roughly speaking, for a stable linear system, a point is reachable with an integral quadratic bound if and only if there is a quadratic Lyapunov function that proves it
(except for points right on the boundary)

RMS gain

recall that the RMS value of a signal is given by

$$\textbf{rms}(z) = \left(\lim_{T\to\infty} \frac{1}{T} \int_0^T \|z(t)\|^2 \, dt \right)^{1/2}$$

assuming the limit exists

now consider a system with input signal u and output signal y

we define its *RMS gain* as the maximum of $\textbf{rms}(y)/\textbf{rms}(u)$, over all u with nonzero RMS value

Lyapunov method for bounding RMS gain

now consider the nonlinear system

$$\dot{x} = f(x,u), \qquad x(0) = 0, \qquad y = g(x,u)$$

with $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^p$

we can use Lyapunov methods to bound its RMS gain

suppose $\gamma \geq 0$, and there is a $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $V(z) \geq 0$ for all z, $V(0) = 0$
- $\dot{V}(z,w) \leq \gamma^2 w^T w - y^T y$ for all $z,\ w$
  ($i.e.$, $\dot{V}(z,w) \leq \gamma^2 w^T w - g(z,w)^T g(z,w)$ for all $z,\ w$)

then, the RMS gain of the system is no more than γ

proof:

$$\begin{aligned} V(x(T)) - V(x(0)) &= \int_0^T \dot{V}(x(t), u(t))\, dt \\ &\leq \int_0^T \left(\gamma^2 u(t)^T u(t) - y(t)^T y(t)\right)\, dt \end{aligned}$$

using $V(x(0)) = V(0) = 0$, $V(x(T)) \geq 0$, we have

$$\int_0^T y(t)^T y(t)\, dt \leq \gamma^2 \int_0^T u(t)^T u(t)\, dt$$

dividing by T and taking the limit $T \to \infty$ yields $\mathbf{rms}(y)^2 \leq \gamma^2 \mathbf{rms}(u)^2$

Bounded-real lemma

let's use a quadratic Lyapunov function $V(z) = z^T P z$ to bound the RMS gain of the stable linear system $\dot{x} = Ax + Bu$, $x(0) = 0$, $y = Cx$

the conditions on V give $P \geq 0$

the condition $\dot{V}(z,w) \leq \gamma^2 w^T w - g(z,w)^T g(z,w)$ becomes

$$\dot{V}(z,w) = 2z^T P(Az + Bw) \leq \gamma^2 w^T w - (Cz)^T Cz$$

for all $z,\ w$

let's write that as a quadratic form in (z,w):

$$\left[\begin{array}{c} z \\ w \end{array}\right]^T \left[\begin{array}{cc} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{array}\right] \left[\begin{array}{c} z \\ w \end{array}\right] \leq 0$$

so we conclude: if there is a $P \geq 0$ such that

$$\left[\begin{array}{cc} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{array}\right] \leq 0$$

then the RMS gain of the linear system is no more than γ

it turns out that for linear systems this condition is not only sufficient, but also necessary

(this result is called the *bounded-real lemma*)

by taking Schur complement, we can express the block 2×2 matrix inequality as

$$A^T P + PA + C^T C + \gamma^{-2} P B B^T P \leq 0$$

(which is a Riccati-like quadratic matrix *inequality* . . .)

Nonlinear optimal control

we consider $\dot{x} = f(x,u)$, $u(t) \in \mathcal{U} \subseteq \mathbf{R}^m$

here we consider u to be an input we can manipulate to achieve some desired response, such as minimizing, or at least making small,

$$J = \int_0^\infty x(t)^T Q x(t)\, dt$$

where $Q \geq 0$

(many other choices for criterion will work) we can solve via dynamic programming: let $V : \mathbf{R}^n \to \mathbf{R}$ denote value function, *i.e.*,

$$V(z) = \min\{J \mid \dot{x} = f(x,u),\ x(0) = z,\ u(t) \in \mathcal{U}\}$$

then the optimal u is given by

$$u^*(t) = \operatorname*{argmin}_{w \in \mathcal{U}} \dot{V}(x(t), w)$$

and with the optimal u we have

$$\dot{V}(x(t), u^*) = -x(t)^T Q x(t)$$

but, it can be very difficult to find V, and therefore u^*

Feedback design via control-Lyapunov functions

suppose there is a function $V : \mathbf{R}^n \to \mathbf{R}$ such that

- $V(z) \geq 0$ for all z
- for all z, $\min_{w \in \mathcal{U}} \dot{V}(z, w) \leq -z^T Q z$

then, the state feedback control law $u(t) = g(x(t))$, with

$$g(z) = \operatorname*{argmin}_{w \in \mathcal{U}} \dot{V}(z, w)$$

results in $J \leq V(x(0))$

in this case V is called a *control-Lyapunov* function for the problem

- if V is the value function, this method recovers the optimal control law

- we've used Lyapunov methods to generate a suboptimal control law, but one with a guaranteed bound on the cost function

- the control law is a greedy one, that simply chooses $u(t)$ to decrease V as quickly as possible (subject to $u(t) \in \mathcal{U}$)

- the inequality $\min_{w \in \mathcal{U}} \dot{V}(z, w) \leq -z^T Q z$ is the inequality form of $\min_{w \in \mathcal{U}} \dot{V}(z, w) = -z^T Q z$, which holds for the optimal input, and V the value function

control-Lyapunov methods offer a good way to generate suboptimal control laws, with performance guarantees, when the optimal control is too hard to find

Lecture 13
Linear matrix inequalities and the S-procedure

- Linear matrix inequalities
- Semidefinite programming
- S-procedure for quadratic forms and quadratic functions

Linear matrix inequalities

suppose $F_0, \ldots, F_n$ are symmetric $m \times m$ matrices

an inequality of the form

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0$$

is called a *linear matrix inequality* (LMI) in the variable $x \in \mathbf{R}^n$

here, $F : \mathbf{R}^n \to \mathbf{R}^{m \times m}$ is an affine function of the variable x

LMIs:

- can represent a wide variety of inequalities
- arise in many problems in control, signal processing, communications, statistics, . . .

most important for us: **LMIs can be solved very efficiently** by newly developed methods (EE364)

"solved" means: we can find x that satisfies the LMI, or determine that no solution exists

Example

$$F(x) = \begin{bmatrix} x_1 + x_2 & x_2 + 1 \\ x_2 + 1 & x_3 \end{bmatrix} \geq 0$$

$$F_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

LMI $F(x) \geq 0$ equivalent to

$$x_1 + x_2 \geq 0, \qquad x_3 \geq 0$$
$$(x_1 + x_2)x_3 - (x_2 + 1)^2 = x_1x_3 + x_2x_3 - x_2^2 - 2x_2 - 1 \geq 0$$

. . . a set of *nonlinear* inequalities in x

Certifying infeasibility of an LMI

- if A, B are symmetric PSD, then $\mathbf{Tr}(AB) \geq 0$:

$$\mathbf{Tr}(AB) = \mathbf{Tr}\left(A^{1/2}B^{1/2}B^{1/2}A^{1/2}\right) = \left\|A^{1/2}B^{1/2}\right\|_F^2$$

- suppose $Z = Z^T$ satisfies

$$Z \geq 0, \qquad \mathbf{Tr}(F_0 Z) < 0, \qquad \mathbf{Tr}(F_i Z) = 0, \quad i = 1, \ldots, n$$

- then if $F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0$,

$$0 \leq \mathbf{Tr}(ZF(x)) = \mathbf{Tr}(ZF_0) < 0$$

  a contradiction

- Z is *certificate* that proves LMI $F(x) \geq 0$ is infeasible

Example: Lyapunov inequality

suppose $A \in \mathbf{R}^{n\times n}$

the *Lyapunov inequality* $A^TP + PA + Q \leq 0$ is an LMI in variable P

meaning: P satisfies the Lyapunov LMI if and only if the quadratic form $V(z) = z^TPz$ satisfies $\dot{V}(z) \leq z^TQz$, for system $\dot{x} = Ax$

the dimension of the variable P is $n(n+1)/2$ (since $P = P^T$)

here, $F(P) = -A^TP - PA - Q$ is affine in P

(we don't need special LMI methods to solve the Lyapunov inequality; we can solve it analytically by solving the Lyapunov equation $A^TP + PA + Q = 0$)

Extensions

multiple LMIs: we can consider multiple LMIs as one, large LMI, by forming block diagonal matrices:

$$F^{(1)}(x) \geq 0, \ldots, F^{(k)}(x) \geq 0 \iff \mathbf{diag}\left(F^{(1)}(x), \ldots, F^{(k)}(x)\right) \geq 0$$

example: we can express a set of linear inequalities as an LMI with diagonal matrices:

$$a_1^T x \leq b_1, \ldots, a_k^T x \leq b_k \iff \mathbf{diag}(b_1 - a_1^T x, \ldots, b_k - a_k^T x) \geq 0$$

linear equality constraints: $a^T x = b$ is the same as the pair of linear inequalities $a^T x \leq b$, $a^T x \geq b$

Example: bounded-real LMI

suppose $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$, $C \in \mathbf{R}^{p \times n}$, and $\gamma > 0$

the *bounded-real LMI* is

$$\left[\begin{array}{cc} A^T P + PA + C^T C & PB \\ B^T P & -\gamma^2 I \end{array} \right] \leq 0, \qquad P \geq 0$$

with variable P

meaning: if P satisfies this LMI, then the quadratic Lyapunov function $V(z) = z^T P z$ proves the RMS gain of the system $\dot{x} = Ax + Bu$, $y = Cx$ is no more than γ

(in fact we can solve this LMI by solving an ARE-like equation, so we don't need special LMI methods . . .)

Strict inequalities in LMIs

sometimes we encounter strict matrix inequalities

$$F(x) \geq 0, \qquad F_{\text{strict}}(x) > 0$$

where F, F_{strict} are affine functions of x

- practical approach: replace $F_{\text{strict}}(x) > 0$ with $F_{\text{strict}}(x) \geq \epsilon I$, where ϵ is small and positive
- if F and F_{strict} are homogenous ($i.e.$, linear functions of x) we can replace with
$$F(x) \geq 0, \qquad F_{\text{strict}}(x) \geq I$$

example: we can replace $A^TP + PA \leq 0$, $P > 0$ (with variable P) with $A^TP + PA \leq 0$, $P \geq I$

Quadratic Lyapunov function for time-varying LDS

we consider time-varying linear system $\dot{x}(t) = A(t)x(t)$ with

$$A(t) \in \{A_1, \ldots, A_K\}$$

- we want to establish some property, such as all trajectories are bounded
- this is hard to do in general (cf. time-invariant LDS)
- let's use quadratic Lyapunov function $V(z) = z^T P z$; we need $P > 0$, and $\dot{V}(z) \leq 0$ for all z, and all possible values of $A(t)$
- gives

$$P > 0, \qquad A_i^T P + P A_i \leq 0, \quad i = 1, \ldots, K$$

- by homogeneity, can write as LMIs

$$P \geq I, \qquad A_i^T P + P A_i \leq 0, \quad i = 1, \ldots, K$$

- in this case V is called *simultaneous Lyapunov function* for the systems $\dot{x} = A_i x$, $i = 1, \ldots, K$

- there is no analytical method ($e.g.$, using AREs) to solve such an LMI, but it is easily done numerically

- if such a P exists, it proves boundedness of trajectories of $\dot{x}(t) = A(t)x(t)$, with

$$A(t) = \theta_1(t)A_1 + \cdots + \theta_K(t)A_K$$

  where $\theta_i(t) \geq 0$, $\theta_1(t) + \cdots + \theta_K(t) = 1$

- in fact, it works for the *nonlinear* system $\dot{x} = f(x)$ provided for each $z \in \mathbf{R}^n$,

$$Df(z) = \theta_1(z)A_1 + \cdots + \theta_K(z)A_K$$

  for some $\theta_i(z) \geq 0$, $\theta_1(z) + \cdots + \theta_K(z) = 1$

Semidefinite programming

a *semidefinite program* (SDP) is an optimization problem with linear objective and LMI and linear equality constraints:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + \cdots + x_n F_n \geq 0 \\ & Ax = b \end{array}$$

most important property for us:
we can solve SDPs globally and efficiently

meaning: we either find a globally optimal solution, or determine that there is no x that satisfies the LMI & equality constraints *example:* let $A \in \mathbf{R}^{n\times n}$ be stable, $Q = Q^T \geq 0$

then the LMI $A^TP + PA + Q \leq 0$, $P \geq 0$ in P means the quadratic Lyapunov function $V(z) = z^TPz$ proves the bound

$$\int_0^\infty x(t)^T Q x(t)\, dt \leq x(0)^T P x(0)$$

now suppose that $x(0)$ is fixed, and we seek the best possible such bound

this can be found by solving the SDP

$$\begin{array}{ll} \text{minimize} & x(0)^T P x(0) \\ \text{subject to} & A^TP + PA + Q \leq 0, \quad P \geq 0 \end{array}$$

with variable P (note that the objective is linear in P)

(in fact we can solve this SDP analytically, by solving the Lyapunov equation)

S-procedure for two quadratic forms

let $F_0 = F_0^T$, $F_1 = F_1^T \in \mathbf{R}^{n\times n}$

when is it true that, for all z, $z^T F_1 z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0$?

in other words, when does nonnegativity of one quadratic form imply nonnegativity of another?

simple condition: there exists $\tau \in \mathbf{R}$, $\tau \geq 0$, with $F_0 \geq \tau F_1$

then for sure we have $z^T F_1 z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0$

(since if $z^T F_1 z \geq 0$, we then have $z^T F_0 z \geq \tau z^T F_1 z \geq 0$)

fact: the converse holds, provided there exists a point u with $u^T F_1 u > 0$

this result is called the *lossless* S-procedure, and is *not* easy to prove

(condition that there exists a point u with $u^T F_1 u > 0$ is called a *constraint qualification*)

S-procedure with strict inequalities

when is it true that, for all z, $z^T F_1 z \geq 0,\ z \neq 0 \ \Rightarrow\ z^T F_0 z > 0$?

in other words, when does nonnegativity of one quadratic form imply positivity of another for nonzero z?

simple condition: suppose there is a $\tau \in \mathbf{R}$, $\tau \geq 0$, with $F_0 > \tau F_1$

fact: the converse holds, provided there exists a point u with $u^T F_1 u > 0$

again, this is *not* easy to prove

Example

let's use quadratic Lyapunov function $V(z) = z^T P z$ to prove stability of

$$\dot{x} = Ax + g(x), \qquad \|g(x)\| \leq \gamma \|x\|$$

we need $P > 0$ and $\dot{V}(x) \leq -\alpha V(x)$ for all x ($\alpha > 0$ is given)

$$\begin{aligned} \dot{V}(x) + \alpha V(x) &= 2x^T P(Ax + g(x)) + \alpha x^T P x \\ &= x^T(A^T P + PA + \alpha P)x + 2x^T P z \\ &= \begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \alpha P & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \end{aligned}$$

where $z = g(x)$

z satisfies $z^T z \leq \gamma^2 x^T x$

so we need $P > 0$ and

$$-\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \alpha P & P \\ P & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0$$

whenever

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \geq 0$$

by S-procedure, this happens if and only if

$$-\begin{bmatrix} A^T P + PA + \alpha P & P \\ P & 0 \end{bmatrix} \geq \tau \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$$

for some $\tau \geq 0$

(constraint qualification holds here)

thus, necessary and sufficient conditions for the existence of quadratic Lyapunov function can be expressed as LMI

$$P > 0, \qquad \begin{bmatrix} A^TP + PA + \alpha P + \tau\gamma^2 I & P \\ P & -\tau I \end{bmatrix} \leq 0$$

in variables P, τ (note condition $\tau \geq 0$ is automatic from $2,2$ block)

by homogeneity, we can write this as

$$P \geq I, \qquad \begin{bmatrix} A^TP + PA + \alpha P + \tau\gamma^2 I & P \\ P & -\tau I \end{bmatrix} \leq 0$$

- solving this LMI to find P is a powerful method
- it beats, for example, solving the Lyapunov equation $A^TP + PA + I = 0$ and hoping the resulting P works

S-procedure for multiple quadratic forms

let $F_0 = F_0^T, \dots, F_k = F_k^T \in \mathbf{R}^{n\times n}$

when is it true that

$$\text{for all } z, \quad z^T F_1 z \geq 0, \dots, z^T F_k z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0 \qquad (1)$$

in other words, when does nonnegativity of a set of quadratic forms imply nonnegativity of another?

simple sufficient condition: suppose there are $\tau_1, \dots, \tau_k \geq 0$, with

$$F_0 \geq \tau_1 F_1 + \cdots + \tau_k F_k$$

then for sure the property (1) above holds

(in this case this is only a sufficient condition; it is not necessary) using the matrix inequality condition

$$F_0 \geq \tau_1 F_1 + \cdots + \tau_k F_k, \qquad \tau_1, \ldots, \tau_k \geq 0$$

as a sufficient condition for

$$\text{for all } z, \quad z^T F_1 z \geq 0, \ldots, z^T F_k z \geq 0 \;\Rightarrow\; z^T F_0 z \geq 0$$

is called the (lossy) S-procedure

the matrix inequality condition is an LMI in $\tau_1, \ldots, \tau_k$, therefore easily solved

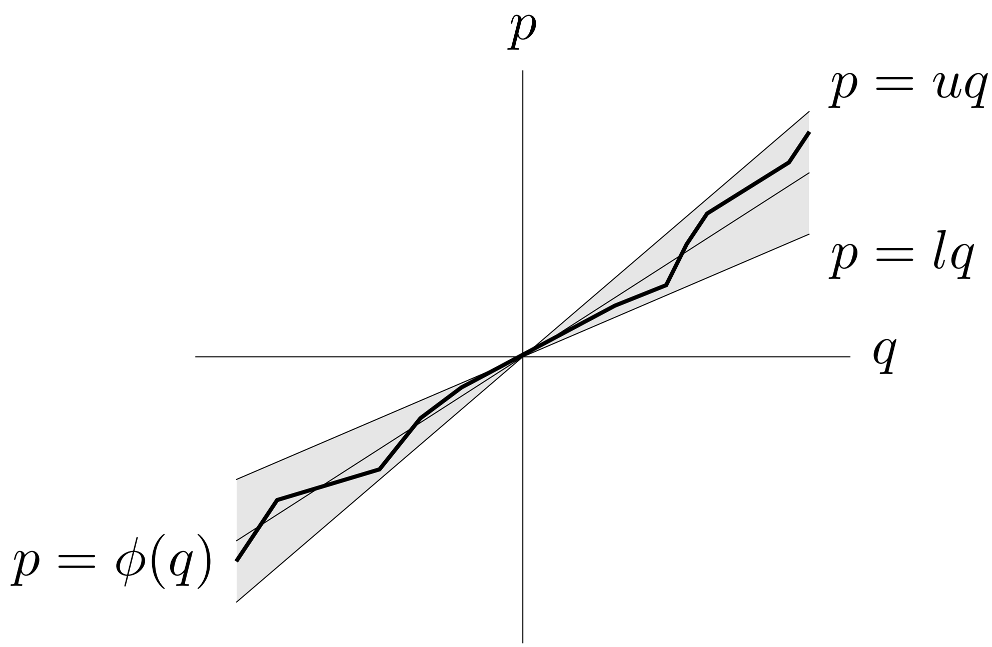
the constants τ_i are called *multipliers*

Lecture 14
Analysis of systems with sector nonlinearities

- Sector nonlinearities
- Lur'e system
- Analysis via quadratic Lyapunov functions
- Extension to multiple nonlinearities

Sector nonlinearities

a function $\phi : \mathbf{R} \to \mathbf{R}$ is said to be in sector $[l, u]$ if for all $q \in \mathbf{R}$, $p = \phi(q)$ lies between lq and uq



can be expressed as quadratic inequality

$$(p - uq)(p - lq) \leq 0 \text{ for all } q, \; p = \phi(q)$$

examples:

- sector $[-1, 1]$ means $|\phi(q)| \leq |q|$
- sector $[0, \infty]$ means $\phi(q)$ and q always have same sign (graph in first & third quadrants)

some equivalent statements:

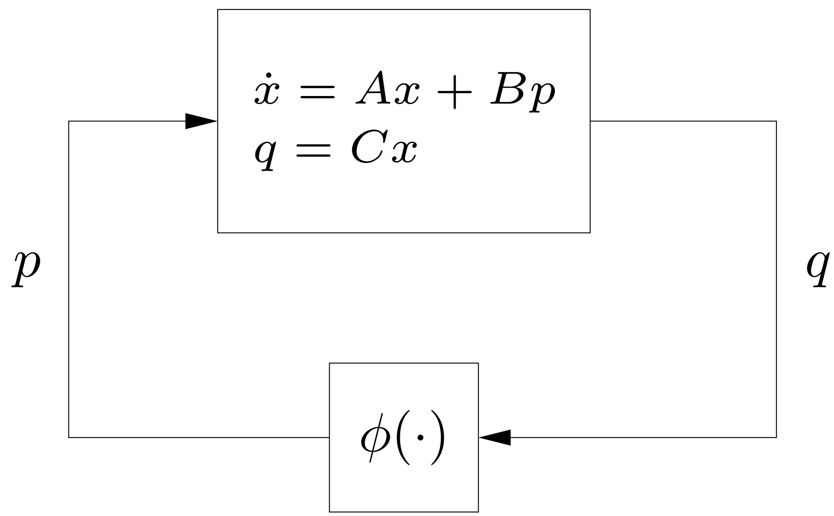
- ϕ is in sector $[l, u]$ iff for all q,

$$\left|\phi(q) - \frac{u+l}{2}q\right| \leq \frac{u-l}{2}|q|$$

- ϕ is in sector $[l, u]$ iff for each q there is $\theta(q) \in [l, u]$ with $\phi(q) = \theta(q)q$

Nonlinear feedback representation

linear dynamical system with nonlinear feedback



closed-loop system: $\dot{x} = Ax + B\phi(Cx)$

- a common representation that separates linear and nonlinear parts
- often p, q are scalar signals

Lur'e system

a (single nonlinearity) *Lur'e system* has the form

$$\dot{x} = Ax + Bp, \qquad q = Cx, \qquad p = \phi(q)$$

where $\phi : \mathbf{R} \to \mathbf{R}$ is in sector $[l, u]$

here A, B, C, l, and u are given; ϕ is otherwise not specified

- a common method for describing nonlinearity and/or uncertainty
- goal is to prove stability, or derive a bound, using only the sector information about ϕ
- if we succeed, the result is strong, since it applies to a large family of nonlinear systems

Stability analysis via quadratic Lyapunov functions

let's try to establish global asymptotic stability of Lur'e system, using quadratic Lyapunov function $V(z) = z^T P z$

we'll require $P > 0$ and $\dot{V}(z) \leq -\alpha V(z)$, where $\alpha > 0$ is given

second condition is:

$$\dot{V}(z) + \alpha V(z) = 2z^T P \left(Az + B\phi(Cz)\right) + \alpha z^T P z \leq 0$$

for all z and all sector $[l, u]$ functions ϕ

same as:

$$2z^T P \left(Az + Bp\right) + \alpha z^T P z \leq 0$$

for all z, and all p satisfying $(p - uq)(p - lq) \leq 0$, where $q = Cz$

we can express this last condition as a quadratic inequality in (z, p):

$$\begin{bmatrix} z \\ p \end{bmatrix}^T \begin{bmatrix} \sigma C^T C & -\nu C^T \\ -\nu C & 1 \end{bmatrix} \begin{bmatrix} z \\ p \end{bmatrix} \leq 0$$

where $\sigma = lu$, $\nu = (l + u)/2$

so $\dot{V} + \alpha V \leq 0$ is equivalent to:

$$\begin{bmatrix} z \\ p \end{bmatrix}^T \begin{bmatrix} A^T P + PA + \alpha P & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} z \\ p \end{bmatrix} \leq 0$$

whenever

$$\begin{bmatrix} z \\ p \end{bmatrix}^T \begin{bmatrix} \sigma C^T C & -\nu C^T \\ -\nu C & 1 \end{bmatrix} \begin{bmatrix} z \\ p \end{bmatrix} \leq 0$$

by (lossless) S-procedure this is equivalent to: there is a $\tau \geq 0$ with

$$\left[\begin{array}{cc} A^TP + PA + \alpha P & PB \\ B^TP & 0 \end{array}\right] \leq \tau \left[\begin{array}{cc} \sigma C^TC & -\nu C^T \\ -\nu C & 1 \end{array}\right]$$

or

$$\left[\begin{array}{cc} A^TP + PA + \alpha P - \tau\sigma C^TC & PB + \tau\nu C^T \\ B^TP + \tau\nu C & -\tau \end{array}\right] \leq 0$$

an LMI in P and τ ($2,2$ block automatically gives $\tau \geq 0$)

by homogeneity, we can replace condition $P > 0$ with $P \geq I$

our final LMI is

$$\left[\begin{array}{cc} A^TP + PA + \alpha P - \tau\sigma C^TC & PB + \tau\nu C^T \\ B^TP + \tau\nu C & -\tau \end{array}\right] \leq 0, \qquad P \geq I$$

with variables P and τ

- hence, can efficiently determine if there exists a quadratic Lyapunov function that proves stability of Lur'e system

- this LMI can also be solved via an ARE-like equation, or by a graphical method that has been known since the 1960s

- this method is more sophisticated and powerful than the 1895 approach:
  - replace nonlinearity with $\phi(q) = \nu q$
  - choose $Q > 0$ ($e.g.$, $Q = I$) and solve Lyapunov equation

$$(A + \nu BC)^T P + P(A + \nu BC) + Q = 0$$

    for P
  - hope P works for nonlinear system

Multiple nonlinearities

we consider system

$$\dot{x} = Ax + Bp, \qquad q = Cx, \qquad p_i = \phi_i(q_i), \quad i = 1, \ldots, m$$

where $\phi_i : \mathbf{R} \to \mathbf{R}$ is sector $[l_i, u_i]$

we seek $V(z) = z^T P z$, with $P > 0$, so that $\dot{V} + \alpha V \leq 0$

last condition equivalent to:

$$\left[\begin{array}{c} z \\ p \end{array}\right]^T \left[\begin{array}{cc} A^T P + PA + \alpha P & PB \\ B^T P & 0 \end{array}\right] \left[\begin{array}{c} z \\ p \end{array}\right] \leq 0$$

whenever

$$(p_i - u_i q_i)(p_i - l_i q_i) \leq 0, \quad i = 1, \ldots, m$$

we can express this last condition as

$$\left[\begin{array}{c} z \\ p \end{array}\right]^T \left[\begin{array}{cc} \sigma c_i^T c_i & -\nu_i c_i^T e_i^T \\ -\nu_i e_i c_i & e_i e_i^T \end{array}\right] \left[\begin{array}{c} z \\ p \end{array}\right] \leq 0, \quad i = 1, \ldots, m$$

where c_i is the ith row of C, e_i is the ith unit vector, $\sigma_i = l_i u_i$, and $\nu_i = (l_i + u_i)/2$

now we use (lossy) S-procedure to get a sufficient condition: there exists $\tau_1, \ldots, \tau_m \geq 0$ such that

$$\left[\begin{array}{cc} A^T P + PA + \alpha P - \sum_{i=1}^m \tau_i \sigma_i c_i^T c_i & PB + \sum_{i=1}^m \tau_i \nu_i c_i^T \\ B^T P + \sum_{i=1}^m \tau_i \nu_i c_i & -\sum_{i=1}^m \tau_i e_i e_i^T \end{array}\right] \leq 0$$

we can write this as:

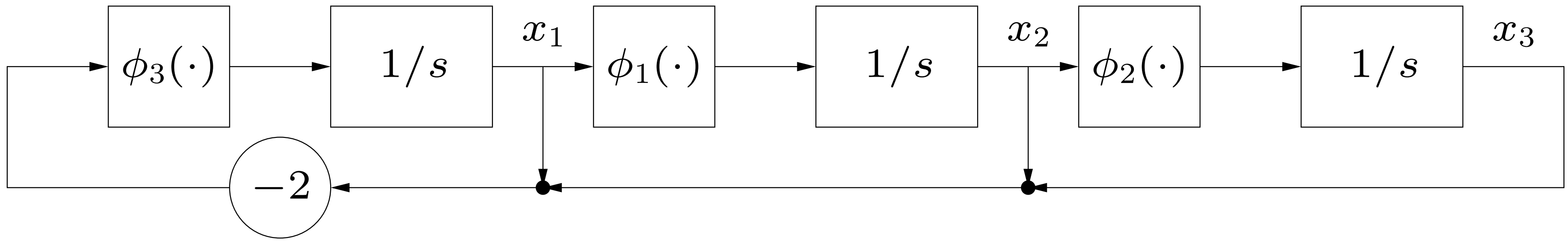
$$\left[\begin{array}{cc} A^TP + PA + \alpha P - C^TDFC & PB + C^TDG \\ B^TP + DGC & -D \end{array}\right] \leq 0$$

where

$$D = \mathbf{diag}(\tau_1, \ldots, \tau_m), \qquad F = \mathbf{diag}(\sigma_1, \ldots, \sigma_m), \qquad G = \mathbf{diag}(\nu_1, \ldots, \nu_m)$$

- this is an LMI in variables P and D
- $2, 2$ block automatically gives us $\tau_i \geq 0$
- by homogeneity, we can add $P \geq I$ to ensure $P > I$
- solving these LMIs allows us to (sometimes) find quadratic Lyapunov functions for Lur'e system with multiple nonlinearities (which was impossible until recently)

Example



we consider system

$$\dot{x}_2 = \phi_1(x_1), \qquad \dot{x}_3 = \phi_2(x_2), \qquad \dot{x}_1 = \phi_3(-2(x_1 + x_2 + x_3))$$

where $\phi_1,\ \phi_2,\ \phi_3$ are sector $[1-\delta, 1+\delta]$

- δ gives the percentage nonlinearity
- for $\delta = 0$, we have (stable) linear system $\dot{x} = \begin{bmatrix} -2 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x$

let's put system in Lur'e form:

$$\dot{x} = Ax + Bp, \qquad q = Cx, \qquad p_i = \phi_i(q_i)$$

where

$$A = 0, \qquad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -2 & -2 \end{bmatrix}$$

the sector limits are $l_i = 1 - \delta$, $u_i = 1 + \delta$

define $\sigma = l_i u_i = 1 - \delta^2$, and note that $(l_i + u_i)/2 = 1$

we take $x(0) = (1, 0, 0)$, and seek to bound $J = \int_0^\infty \|x(t)\|^2 \, dt$

(for $\delta = 0$ we can calculate J exactly by solving a Lyapunov equation)

we'll use quadratic Lyapunov function $V(z) = z^T P z$, with $P \geq 0$

Lyapunov conditions for bounding J: if $\dot{V}(z) \leq -z^T z$ whenever the sector conditions are satisfied, then $J \leq x(0)^T P x(0) = P_{11}$

use S-procedure as above to get sufficient condition:

$$\begin{bmatrix} A^T P + PA + I - \sigma C^T D C & PB + C^T D \\ B^T P + DC & -D \end{bmatrix} \leq 0$$

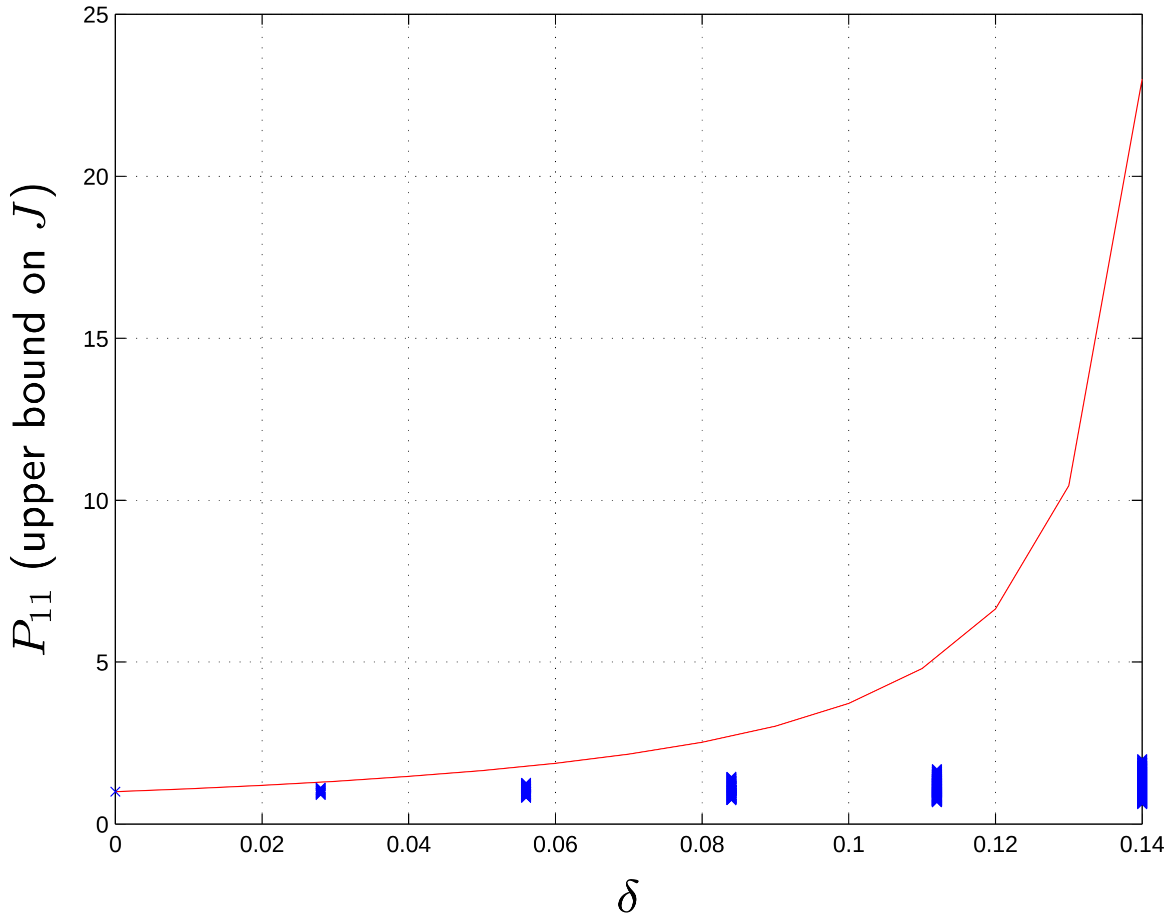
which is an LMI in variables P and $D = \mathbf{diag}(\tau_1, \tau_2, \tau_3)$

note that LMI gives $\tau_i \geq 0$ automatically to get best bound on J for given δ, we solve SDP

$$\begin{array}{ll} \text{minimize} & P_{11} \\ \text{subject to} & \left[\begin{array}{cc} A^TP + PA + I - \sigma C^TDC & PB + C^TD \\ B^TP + DC & -D \end{array}\right] \leq 0 \\ & P \geq 0 \end{array}$$

with variables P and D (which is diagonal)

optimal value gives best bound on J that can be obtained from a quadratic Lyapunov function, using S-procedure



- top plot shows bound on J; bottom points show results for constant linear ϕ_i's chosen at random in interval $1 \pm \delta$
- bound is exact for $\delta = 0$; for $\delta \geq 0.15$, LMI is infeasible

Lecture 15
Perron-Frobenius Theory

- Positive and nonnegative matrices and vectors
- Perron-Frobenius theorems
- Markov chains
- Economic growth
- Population dynamics
- Max-min and min-max characterization
- Power control
- Linear Lyapunov functions
- Metzler matrices

Positive and nonnegative vectors and matrices

we say a matrix or vector is

- *positive* (or *elementwise positive*) if all its entries are positive
- *nonnegative* (or *elementwise nonnegative*) if all its entries are nonnegative

we use the notation $x > y$ ($x \geq y$) to mean $x - y$ is elementwise positive (nonnegative)

warning: if A and B are square and symmetric, $A \geq B$ can mean:

- $A - B$ is PSD ($i.e.$, $z^T Az \geq z^T Bz$ for all z), or
- $A - B$ elementwise positive ($i.e.$, $A_{ij} \geq B_{ij}$ for all $i,\ j$)

in this lecture, $>$ and $\geq$ mean elementwise

Application areas

nonnegative matrices arise in many fields, $e.g.$,

- economics
- population models
- graph theory
- Markov chains
- power control in communications
- Lyapunov analysis of large scale systems

Basic facts

if $A \geq 0$ and $z \geq 0$, then we have $Az \geq 0$

conversely: if for all $z \geq 0$, we have $Az \geq 0$, then we can conclude $A \geq 0$

in other words, matrix multiplication preserves nonnegativity if and only if the matrix is nonnegative

if $A > 0$ and $z \geq 0$, $z \neq 0$, then $Az > 0$

conversely, if whenever $z \geq 0$, $z \neq 0$, we have $Az > 0$, then we can conclude $A > 0$

if $x \geq 0$ and $x \neq 0$, we refer to $d = (1/\mathbf{1}^T x)x$ as its *distribution* or normalized form

$d_i = x_i / (\sum_j x_j)$ gives the fraction of the total of x, given by x_i

Regular nonnegative matrices

suppose $A \in \mathbf{R}^{n \times n}$, with $A \geq 0$

A is called *regular* if for some $k \geq 1$, $A^k > 0$

meaning: form directed graph on nodes $1, \ldots, n$, with an arc from j to i whenever $A_{ij} > 0$

then $(A^k)_{ij} > 0$ if and only if there is a path of length k from j to i

A is regular if for some k there is a path of length k from every node to every other node examples:

- any positive matrix is regular
- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are not regular
- $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is regular

Perron-Frobenius theorem for regular matrices

suppose $A \in \mathbf{R}^{n \times n}$ is nonnegative and regular, $i.e.$, $A^k > 0$ for some k

then

- there is an eigenvalue $\lambda_{\rm pf}$ of A that is real and positive, with positive left and right eigenvectors
- for any other eigenvalue λ, we have $|\lambda| < \lambda_{\rm pf}$
- the eigenvalue $\lambda_{\rm pf}$ is simple, $i.e.$, has multiplicity one, and corresponds to a 1×1 Jordan block

the eigenvalue $\lambda_{\rm pf}$ is called the *Perron-Frobenius* (PF) eigenvalue of A

the associated positive (left and right) eigenvectors are called the (left and right) PF eigenvectors (and are unique, up to positive scaling)

Perron-Frobenius theorem for nonnegative matrices

suppose $A \in \mathbf{R}^{n \times n}$ and $A \geq 0$

then

- there is an eigenvalue $\lambda_{\rm pf}$ of A that is real and nonnegative, with associated nonnegative left and right eigenvectors
- for any other eigenvalue λ of A, we have $|\lambda| \leq \lambda_{\rm pf}$

$\lambda_{\rm pf}$ is called the *Perron-Frobenius* (PF) eigenvalue of A

the associated nonnegative (left and right) eigenvectors are called (left and right) PF eigenvectors

in this case, they need not be unique, or positive

Markov chains

we consider stochastic process $X(0),\ X(1), \ldots$ with values in $\{1, \ldots, n\}$

$\mathbf{Prob}(X(t+1) = i | X(t) = j) = P_{ij}$

P is called the *transition matrix*; clearly $P_{ij} \geq 0$

let $p(t) \in \mathbf{R}^n$ be the distribution of $X(t)$, *i.e.*, $p_i(t) = \mathbf{Prob}(X(t) = i)$

then we have $p(t+1) = Pp(t)$

note: standard notation uses transpose of P, and row vectors for probability distributions

P is a *stochastic matrix*, *i.e.*, $P \geq 0$ and $\mathbf{1}^T P = \mathbf{1}^T$

so $\mathbf{1}$ is a left eigenvector with eigenvalue 1, which is in fact the PF eigenvalue of P

Equilibrium distribution

let π denote a PF (right) eigenvector of P, with $\pi \geq 0$ and $\mathbf{1}^T\pi = 1$

since $P\pi = \pi$, π corresponds to an *invariant distribution* or *equilibrium distribution* of the Markov chain

now suppose P is regular, which means for some k, $P^k > 0$

since $(P^k)_{ij}$ is $\mathbf{Prob}(X(t+k) = i|X(t) = j)$, this means there is positive probability of transitioning from any state to any other in k steps

since P is regular, there is a unique invariant distribution π, which satisfies $\pi > 0$

the eigenvalue 1 is simple and dominant, so we have $p(t) \to \pi$, no matter what the initial distribution $p(0)$

in other words: the distribution of a regular Markov chain always converges to the unique invariant distribution

Rate of convergence to equilibrium distribution

rate of convergence to equilibrium distribution depends on second largest eigenvalue magnitude, $i.e.$,

$$\mu = \max\{|\lambda_2|, \ldots, |\lambda_n|\}$$

where λ_i are the eigenvalues of P, and $\lambda_1 = \lambda_{\rm pf} = 1$

(μ is sometimes called the SLEM of the Markov chain)

the *mixing time* of the Markov chain is given by

$$T = \frac{1}{\log(1/\mu)}$$

(roughly, number of steps over which deviation from equilibrium distribution decreases by factor e)

Dynamic interpretation

consider $x(t+1) = Ax(t)$, with $A \geq 0$ and regular

then by PF theorem, $\lambda_{\rm pf}$ is the unique dominant eigenvalue

let $v,\ w > 0$ be the right and left PF eigenvectors of A, with $\mathbf{1}^T v = 1$, $w^T v = 1$

then as $t \to \infty$, $(\lambda_{\rm pf}^{-1} A)^t \to v w^T$

for any $x(0) \geq 0$, $x(0) \neq 0$, we have

$$\frac{1}{\mathbf{1}^T x(t)} x(t) \to v$$

as $t \to \infty$, *i.e.*, the distribution of $x(t)$ converges to v

we also have $x_i(t+1)/x_i(t) \to \lambda_{\rm pf}$, *i.e.*, the one-period growth factor in each component always converges to $\lambda_{\rm pf}$

Economic growth

we consider an economy, with activity level $x_i \geq 0$ in sector i, $i = 1, \ldots, n$

given activity level x in period t, in period $t+1$ we have $x(t+1) = Ax(t)$, with $A \geq 0$

$A_{ij} \geq 0$ means activity in sector j does not decrease activity in sector i, $i.e.$, the activities are mutually noninhibitory

we'll assume that A is regular, with PF eigenvalue λ_{pf}, and left and right PF eigenvectors $w,\ v$, with $\mathbf{1}^T v = 1$, $w^T v = 1$

PF theorem tells us:

- $x_i(t+1)/x_i(t)$, the growth factor in sector i over the period from t to $t+1$, each converge to λ_{pf} as $t \rightarrow \infty$
- the distribution of economic activity ($i.e.$, x normalized) converges to v

- asymptotically the economy exhibits (almost) balanced growth, by the factor $\lambda_{\rm pf}$, in each sector

these hold independent of the original economic activity, provided it is nonnegative and nonzero

what does left PF eigenvector w mean?

for large t we have

$$x(t) \sim \lambda_{\rm pf}^t w^T x(0) v$$

where $\sim$ means we have dropped terms small compared to dominant term

so asymptotic economic activity is scaled by $w^T x(0)$

in particular, w_i gives the relative *value* of activity i in terms of long term economic activity

Population model

$x_i(t)$ denotes number of individuals in group i at period t

groups could be by age, location, health, marital status, etc.

population dynamics is given by $x(t+1) = Ax(t)$, with $A \geq 0$

A_{ij} gives the fraction of members of group j that move to group i, or the number of members in group i created by members of group j ($e.g.$, in births)

$A_{ij} \geq 0$ means the more we have in group j in a period, the more we have in group i in the next period

- if $\sum_i A_{ij} = 1$, population is preserved in transitions out of group j
- we can have $\sum_i A_{ij} > 1$, if there are births (say) from members of group j
- we can have $\sum_i A_{ij} < 1$, if there are deaths or attrition in group j

now suppose A is regular

- PF eigenvector v gives asymptotic population distribution
- PF eigenvalue $\lambda_{\rm pf}$ gives asymptotic growth rate (if > 1) or decay rate (if < 1)
- $w^T x(0)$ scales asymptotic population, so w_i gives relative value of initial group i to long term population

Path count in directed graph

we have directed graph on n nodes, with adjacency matrix $A \in \mathbf{R}^{n \times n}$

$$A_{ij} = \begin{cases} 1 & \text{there is an edge from node } j \text{ to node } i \\ 0 & \text{otherwise} \end{cases}$$

$\left(A^k\right)_{ij}$ is number of paths from j to i of length k

now suppose A is regular

then for large k,

$$A^k \sim \lambda_{\mathrm{pf}}^k v w^T = \lambda_{\mathrm{pf}}^k (\mathbf{1}^T w) v (w/\mathbf{1}^T w)^T$$

($\sim$ means: keep only dominant term)

$v,\ w$ are right, left PF eigenvectors, normalized as $\mathbf{1}^T v = 1$, $w^T v = 1$

total number of paths of length k: $\mathbf{1}^T A^k \mathbf{1} \approx \lambda_{\mathrm{pf}}^k (\mathbf{1}^T w)$

for k large, we have (approximately)

- λ_{pf} is factor of increase in number of paths when length increases by one
- v_i: fraction of length k paths that end at i
- $w_j/\mathbf{1}^T w$: fraction of length k paths that start at j
- $v_i w_j/\mathbf{1}^T w$: fraction of length k paths that start at j, end at i

- v_i measures importance/connectedness of node i as a *sink*
- $w_j/\mathbf{1}^T w$ measures importance/connectedness of node j as a *source*

(Part of) proof of PF theorem for positive matrices

suppose $A > 0$, and consider the optimization problem

$$\begin{array}{ll} \text{maximize} & \delta \\ \text{subject to} & Ax \geq \delta x \text{ for some } x \geq 0, \quad x \neq 0 \end{array}$$

note that we can assume $\mathbf{1}^T x = 1$

interpretation: with $y_i = (Ax)_i$, we can interpret y_i/x_i as the 'growth factor' for component i

problem above is to find the input distribution that maximizes the minimum growth factor

let λ_0 be the optimal value of this problem, and let v be an optimal point, $i.e.$, $v \geq 0$, $v \neq 0$, and $Av \geq \lambda_0 v$

we will show that λ_0 is the PF eigenvalue of A, and v is a PF eigenvector

first let's show $Av = \lambda_0 v$, *i.e.*, v is an eigenvector associated with λ_0

if not, suppose that $(Av)_k > \lambda_0 v_k$

now let's look at $\tilde{v} = v + \epsilon e_k$

we'll show that for small $\epsilon > 0$, we have $A\tilde{v} > \lambda_0 \tilde{v}$, which means that $A\tilde{v} \geq \delta \tilde{v}$ for some $\delta > \lambda_0$, a contradiction

for $i \neq k$ we have

$$(A\tilde{v})_i = (Av)_i + A_{ik}\epsilon > (Av)_i \geq \lambda_0 v_i = \lambda_0 \tilde{v}_i$$

so for any $\epsilon > 0$ we have $(A\tilde{v})_i > \lambda_0 \tilde{v}_i$

$$\begin{aligned} (A\tilde{v})_k - \lambda_0 \tilde{v}_k &= (Av)_k + A_{kk}\epsilon - \lambda_0 v_k - \lambda_0 \epsilon \\ &= (Av)_k - \lambda_0 v_k - \epsilon(\lambda_0 - A_{kk}) \end{aligned}$$

since $(Av)_k - \lambda_0 v_k > 0$, we conclude that for small $\epsilon > 0$, $(A\tilde{v})_k - \lambda_0 \tilde{v}_k > 0$

to show that $v > 0$, suppose that $v_k = 0$

from $Av = \lambda_0 v$, we conclude $(Av)_k = 0$, which contradicts $Av > 0$ (which follows from $A > 0$, $v \geq 0$, $v \neq 0$)

now suppose $\lambda \neq \lambda_0$ is another eigenvalue of A, *i.e.*, $Az = \lambda z$, where $z \neq 0$

let $|z|$ denote the vector with $|z|_i = |z_i|$

since $A \geq 0$ we have $A|z| \geq |Az| = |\lambda||z|$

from the definition of λ_0 we conclude $|\lambda| \leq \lambda_0$

(to show strict inequality is harder)

Max-min ratio characterization

proof shows that PF eigenvalue is optimal value of optimization problem

$$\begin{array}{ll} \text{maximize} & \min_i \frac{(Ax)_i}{x_i} \\ \text{subject to} & x > 0 \end{array}$$

and that PF eigenvector v is optimal point:

- PF eigenvector v maximizes the minimum growth factor over components
- with optimal v, growth factors in all components are equal (to λ_{pf})

in other words: by maximizing minimum growth factor, we actually achieve balanced growth

Min-max ratio characterization

a related problem is

$$\begin{array}{ll} \text{minimize} & \max_i \frac{(Ax)_i}{x_i} \\ \text{subject to} & x > 0 \end{array}$$

here we seek to minimize the maximum growth factor in the coordinates

the solution is surprising: the optimal value is $\lambda_{\rm pf}$ and the optimal x is the PF eigenvector v

- if A is nonnegative and regular, and $x > 0$, the n growth factors $(Ax)_i/x_i$ 'straddle' $\lambda_{\rm pf}$: at least one is $\geq \lambda_{\rm pf}$, and at least one is $\leq \lambda_{\rm pf}$
- when we take x to be the PF eigenvector v, all the growth factors are equal, and solve both max-min and min-max problems

Power control

we consider n transmitters with powers $P_1, \ldots, P_n > 0$, transmitting to n receivers

path gain from transmitter j to receiver i is $G_{ij} > 0$

signal power at receiver i is $S_i = G_{ii}P_i$

interference power at receiver i is $I_i = \sum_{k \neq i} G_{ik}P_k$

signal to interference ratio (SIR) is

$$S_i/I_i = \frac{G_{ii}P_i}{\sum_{k \neq i} G_{ik}P_k}$$

how do we set transmitter powers to maximize the minimum SIR?

we can just as well minimize the maximum interference to signal ratio, *i.e.*, solve the problem

$$\begin{array}{ll} \text{minimize} & \max_i \frac{(\tilde{G}P)_i}{P_i} \\ \text{subject to} & P > 0 \end{array}$$

where

$$\tilde{G}_{ij} = \begin{cases} G_{ij}/G_{ii} & i \neq j \\ 0 & i = j \end{cases}$$

since $\tilde{G}^2 > 0$, $\tilde{G}$ is regular, so solution is given by PF eigenvector of $\tilde{G}$

PF eigenvalue λ_{pf} of $\tilde{G}$ is the optimal interference to signal ratio, *i.e.*, maximum possible minimum SIR is $1/\lambda_{\text{pf}}$

with optimal power allocation, all SIRs are equal

note: $\tilde{G}$ is the matrix of ratios of interference to signal path gains

Nonnegativity of resolvent

suppose A is nonnegative, with PF eigenvalue $\lambda_{\rm pf}$, and $\lambda \in \mathbf{R}$

then $(\lambda I - A)^{-1}$ exists and is nonnegative, if and only if $\lambda > \lambda_{\rm pf}$

for any square matrix A the power series expansion

$$(\lambda I - A)^{-1} = \frac{1}{\lambda} I + \frac{1}{\lambda^2} A + \frac{1}{\lambda^3} A^2 + \cdots$$

converges provided $|\lambda|$ is larger than all eigenvalues of A

if $\lambda > \lambda_{\rm pf}$, this shows that $(\lambda I - A)^{-1}$ is nonnegative

to show converse, suppose $(\lambda I - A)^{-1}$ exists and is nonnegative, and let $v \neq 0$, $v \geq 0$ be a PF eigenvector of A

then we have

$$(\lambda I - A)^{-1} v = \frac{1}{\lambda - \lambda_{\rm pf}} v \geq 0$$

and it follows that $\lambda > \lambda_{\rm pf}$

Equilibrium points

consider $x(t+1) = Ax(t) + b$, where A and b are nonnegative

equilibrium point is given by $x_{\rm eq} = (I - A)^{-1} b$

by resolvent result, if A is stable, then $(I - A)^{-1}$ is nonnegative, so equilibrium point $x_{\rm eq}$ is nonnegative for any nonnegative b

moreover, equilibrium point is monotonic function of b: for $\tilde{b} \geq b$, we have $\tilde{x}_{\rm eq} \geq x_{\rm eq}$

conversely, if system has a nonnegative equilibrium point, for every nonnegative choice of b, then we can conclude A is stable

Iterative power allocation algorithm

we consider again the power control problem

suppose γ is the desired or target SIR

simple iterative algorithm: at each step t,

1. first choose $\tilde{P}_i$ so that

$$\frac{G_{ii}\tilde{P}_i}{\sum_{k\neq i} G_{ik}P_k(t)} = \gamma$$

   $\tilde{P}_i$ is the transmit power that would make the SIR of receiver i equal to γ, *assuming none of the other powers change*

2. set $P_i(t+1) = \tilde{P}_i + \sigma_i$, where $\sigma_i > 0$ is a parameter $i.e.$, add a little extra power to each transmitter)

each receiver only needs to know its current SIR to adjust its power: if current SIR is α dB below (above) γ, then increase (decrease) transmitter power by α dB, then add the extra power σ

$i.e.$, this is a *distributed algorithm*

question: does it work? (we assume that $P(0) > 0$)

answer: yes, if and only if γ is less than the maximum achievable SIR, $i.e.$, $\gamma < 1/\lambda_{\mathrm{pf}}(\tilde{G})$

to see this, algorithm can be expressed as follows:

- in the first step, we have $\tilde{P} = \gamma \tilde{G} P(t)$

- in the second step we have $P(t+1) = \tilde{P} + \sigma$

and so we have

$$P(t+1) = \gamma \tilde{G} P(t) + \sigma$$

a linear system with constant input

PF eigenvalue of $\gamma\tilde{G}$ is $\gamma\lambda_{\rm pf}$, so linear system is stable if and only if $\gamma\lambda_{\rm pf} < 1$

power converges to equilibrium value

$$P_{\rm eq} = (I - \gamma\tilde{G})^{-1}\sigma$$

(which is positive, by resolvent result)

now let's show this equilibrium power allocation achieves SIR at least γ for each receiver

we need to verify $\gamma\tilde{G}P_{\rm eq} \leq P_{\rm eq}$, *i.e.*,

$$\gamma\tilde{G}(I - \gamma\tilde{G})^{-1}\sigma \leq (I - \gamma\tilde{G})^{-1}\sigma$$

or, equivalently,

$$(I - \gamma\tilde{G})^{-1}\sigma - \gamma\tilde{G}(I - \gamma\tilde{G})^{-1}\sigma \geq 0$$

which holds, since the lefthand side is just σ

Linear Lyapunov functions

suppose $A \geq 0$

then $\mathbf{R}^n_+$ is invariant under system $x(t+1) = Ax(t)$

suppose $c > 0$, and consider the linear Lyapunov function $V(z) = c^T z$

if $V(Az) \leq \delta V(z)$ for some $\delta < 1$ and all $z \geq 0$, then V proves (nonnegative) trajectories converge to zero

fact: a nonnegative regular system is stable if and only if there is a linear Lyapunov function that proves it

to show the 'only if' part, suppose A is stable, *i.e.*, $\lambda_{\rm pf} < 1$

take $c = w$, the (positive) left PF eigenvector of A

then we have $V(Az) = w^T Az = \lambda_{\rm pf} w^T z$, *i.e.*, V proves all nonnegative trajectories converge to zero

Weighted ℓ_1-norm Lyapunov function

to make the analysis apply to *all* trajectories, we can consider the weighted sum absolute value (or weighted ℓ_1-norm) Lyapunov function

$$V(z) = \sum_{i=1}^{n} w_i |z_i| = w^T |z|$$

then we have

$$V(Az) = \sum_{i=1}^{n} w_i |(Az)_i| \leq \sum_{i=1}^{n} w_i (A|z|)_i = w^T A|z| = \lambda_{\mathrm{pf}} w^T |z|$$

which shows that V decreases at least by the factor λ_{pf}

conclusion: a nonnegative regular system is stable if and only if there is a weighted sum absolute value Lyapunov function that proves it

SVD analysis

suppose $A \in \mathbf{R}^{m \times n}$, $A \geq 0$

then $A^T A \geq 0$ and $AA^T \geq 0$ are nonnegative

hence, there are nonnegative left & right singular vectors v_1, w_1 associated with σ_1

in particular, there is an optimal rank-1 approximation of A that is nonnegative

if $A^T A$, AA^T are regular, then we conclude

- $\sigma_1 > \sigma_2$, $i.e.$, maximum singular value is isolated
- associated singular vectors are positive: $v_1 > 0$, $w_1 > 0$

Continuous time results

we have already seen that $\mathbf{R}_+^n$ is invariant under $\dot{x} = Ax$ if and only if $A_{ij} \geq 0$ for $i \neq j$

such matrices are called *Metzler matrices*

for a Metzler matrix, we have

- there is an eigenvalue $\lambda_{\rm metzler}$ of A that is real, with associated nonnegative left and right eigenvectors
- for any other eigenvalue λ of A, we have $\Re\lambda \leq \lambda_{\rm metzler}$

  $i.e.$, the eigenvalue $\lambda_{\rm metzler}$ is dominant for system $\dot{x} = Ax$
- if $\lambda > \lambda_{\rm metzler}$, then $(\lambda I - A)^{-1} \geq 0$

the analog of the stronger Perron-Frobenius results:

if $(\tau I + A)^k > 0$, for some τ and some k, then

- the left and right eigenvectors associated with eigenvalue $\lambda_{\mathrm{metzler}}$ of A are positive
- for any other eigenvalue λ of A, we have $\Re\lambda < \lambda_{\mathrm{metzler}}$

  i.e., the eigenvalue $\lambda_{\mathrm{metzler}}$ is strictly dominant for system $\dot{x} = Ax$

Derivation from Perron-Frobenius Theory

suppose A is Metzler, and choose τ s.t. $\tau I + A \geq 0$
($e.g.$, $\tau = 1 - \min_i A_{ii}$)

by PF theory, $\tau I + A$ has PF eigenvalue $\lambda_{\rm pf}$, with associated right and left eigenvectors $v \geq 0$, $w \geq 0$

from $(\tau I + A)v = \lambda_{\rm pf} v$ we get $Av = (\lambda_{\rm pf} - \tau)v = \lambda_0 v$, and similarly for w

we'll show that $\Re\lambda \leq \lambda_0$ for any eigenvalue λ of A

suppose λ is an eigenvalue of A

suppose $\tau + \lambda$ is an eigenvalue of $\tau I + A$

by PF theory, we have $|\tau + \lambda| \leq \lambda_{\rm pf} = \tau + \lambda_0$

this means λ lies inside a circle, centered at $-\tau$, that passes through λ_0

which implies $\Re\lambda \leq \lambda_0$

Linear Lyapunov function

suppose $\dot{x} = Ax$ is stable, and A is Metzler, with $(\tau I + A)^k > 0$ for some τ and some k

we can show that all nonnegative trajectories converge to zero using a linear Lyapunov function

let $w > 0$ be left eigenvector associated with dominant eigenvalue $\lambda_{\mathrm{metzler}}$

then with $V(z) = w^T z$ we have

$$\dot{V}(z) = w^T A z = \lambda_{\mathrm{metzler}} w^T z = \lambda_{\mathrm{metzler}} V(z)$$

since $\lambda_{\mathrm{metzler}} < 0$, this proves $w^T z \to 0$